Amplitude Equations for Electrostatic Waves: universal singular behavior in the limit of weak instability

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Abstract

An amplitude equation for an unstable mode in a collisionless plasma is derived from the dynamics on the unstable manifold of the equilibrium $F_0(v)$. The mode eigenvalue arises from a simple zero of the dielectric $\epsilon_k(z)$; as the linear growth rate γ vanishes, the eigenvalue merges with the continuous spectrum on the imaginary axis and disappears. The evolution of the mode amplitude $\rho(t)$ is studied using an expansion in ρ. As $\gamma \to 0^+$, the expansion coefficients diverge, but these singularities are absorbed by rescaling the amplitude: $\rho(t) \equiv \gamma^2\, r(\gamma t)$. This renders the theory finite and also indicates that the electric field exhibits trapping scaling $E \sim \gamma^2$. These singularities and scalings are independent of the specific $F_0(v)$ considered. The asymptotic dynamics of $r(\tau)$ can depend on F_0 only through $\exp i\xi$ where $d\epsilon_k/dz = |\epsilon_k'| \exp -i\xi/2$. Similar results also hold for the electric field and distribution function.

I. INTRODUCTION

The evolution of an unstable electrostatic mode is a fundamental problem in collisionless plasma theory. Although quite idealized, this evolution involves many features that are essential to more complicated and realistic problems, in particular there is a singular interaction between the wave and resonant particles. This resonance drives the initial growth of the unstable linear mode, and then trapping of resonant particles by the finite amplitude wave marks the onset of strong nonlinear effects that saturate the instability. These nonlinear effects are difficult to treat analytically while maintaining the self-consistent relationship between electric field and particles, and calculations on this problem have emphasized special regimes which allow simplifying approximations, e.g. a "bump on tail" distribution or instability driven by a small cold beam. [1–11]

In this paper, I describe a new approach which simplifies the problem by restricting attention to the dynamics occuring on the unstable manifold of the equilibrium. Physically this restriction means I consider initial conditions in which *only* the unstable modes are excited, rather than allowing arbitrary initial conditions comprised of all linear modes. Mathematically the unstable manifold is finite-dimensional and this reduction in dimension provides a considerable simplification.

The restriction on initial conditions is compensated by the freedom from inessential assumptions about the equilibrium $F_0(v,\mu)$. Here μ denotes any parameters such as density or temperature that determine the properties of F_0; it is not necessary to make a specific choice for μ, rather we let $F_0(v,\mu_c)$ denote the critical equilibrium. For $\mu<\mu_c$, $F_0(v,\mu)$ is linearly stable and for $\mu>\mu_c$ there is an unstable mode (or modes) with linear growth rate γ. The limit $\mu\to\mu_c$ from the unstable regime will usually be denoted $\gamma\to 0^+$. Thus I am able to give a unified treatment of instabilities in beam-plasma systems with warm or cold beams as well as two-stream instabilities for equilibria with counterstreaming components of equal density.

An additional motivation for this approach is the possibility that the dynamics on the

manifold will exhibit universal features of the instability. This hope arises from experience with simpler bifurcations in dissipative dynamical systems. Consider, for example, a Hopf bifurcation [12,13] in which an equilibrium loses stability as an isolated conjugate pair of eigenvalues (λ, λ^*) cross the imaginary axis while all other modes remain stable. In this situation, there is a two-dimensional unstable manifold associated with the unstable modes, and the isolation of critical eigenvalues permits the dynamics near the equilibrium to be rigorously reduced to the two-dimensional dynamics on the unstable manifold. In polar variables (ρ, θ), this reduced two-dimensional system has the form

$$\dot{\rho} = \gamma\rho + a_1\rho^3 + a_2\rho^5 + \mathcal{O}(\rho^7) \tag{1}$$

$$\dot{\theta} = \omega + a_1'\rho^2 + a_2'\rho^4 + \mathcal{O}(\rho^6) \tag{2}$$

where $A(t) = \rho(t)\, e^{-i\theta(t)}$ is the amplitude of the unstable linear mode and $\lambda = \gamma - i\omega$ is the critical eigenvalue; the evolution of $\rho(t)$ decouples from the phase $\theta(t)$ so that the dynamics can be easily analyzed. Provided the cubic coefficient a_1 is non-zero at the onset of instability, these equations have a universal structure in the limit of weak instability $\gamma \to 0^+$. More specifically, by setting $\rho(t) = \sqrt{\gamma}\, r(\gamma t)$ we obtain from (1)

$$\frac{dr}{d\tau} = r + a_1 r^3 + \gamma a_2 r^5 + \mathcal{O}(\gamma^2) \tag{3}$$

where $\tau = \gamma t$. As $\gamma \to 0^+$, the terms of higher order in r vanish leaving $\dot{r} = r + a_1 r^3$, an asymptotic equation of universal form reflecting the specific problem under consideration only through the coefficient a_1. In this way, the unstable manifold dynamics for Hopf bifurcation reveals the slow time scale τ and, more interestingly, a universal scaling behavior $\rho \sim \sqrt{\gamma}$ for the mode amplitude.

There are many key differences between such a simple dissipative bifurcation and the bifurcation arising from the appearance of an unstable mode in the linear spectrum of a Vlasov equilibrium. The Vlasov equation is a Hamiltonian dynamical system and the spectrum for a stable equilibrium is pure imaginary. [14–16] The eigenvalues describing the unstable modes are not isolated at the onset of instability, in fact they appear in the

spectrum for the first time at onset and are embedded in the continuous spectrum on the imaginary axis. [17] Furthermore, without dissipation, one does not expect the unstable manifold associated with the instability to be attracting.

Despite these important differences, it is feasible to adapt the derivation of (1) - (2) to the bifurcation of an electrostatic mode and to obtain the corresponding equations for the dynamics on the unstable manifold. The $\gamma \to 0^+$ limit of these equations is remarkable and provides a striking contrast with the familiar limiting behavior in Hopf bifurcation. [18] For an electrostatic mode, the coefficients (a_j, a'_j) are singular as $\gamma \to 0^+$ with asymptotic form $\gamma^{-(4j-1)}$. These divergences are not unphysical, rather they signal the presence of a different scaling from that characterizing a Hopf bifurcation; by setting $\rho(t) = \gamma^2 r(\gamma t)$ one can absorb the singular behavior at every order leaving rescaled equations for r that are well behaved as $\gamma \to 0^+$. Unlike the rescaled equation for Hopf bifurcation (3), in the Vlasov case the terms that are higher order in r are *not* higher order in γ, rather at each order $b_j\, r^{2j}$ the rescaled coefficient $b_j = \gamma^{(4j-1)}\, a_j$ is order unity as $\gamma \to 0^+$. Hence the equation for $dr/d\tau$ does not truncate, and the $\gamma \to 0^+$ equation retains an infinite set of terms. The first two coefficients have been calculated and shown to be *independent* of $F_0(v, \mu_c)$ as $\gamma \to 0^+$ with values $b_1 = -1/4$ and $b_2 = 13/64$. The coefficients at higher order are not explicitly determined, however one can prove that at each order there is a universal function Q_j, independent of $F_0(v, \mu_c)$, such that as $\gamma \to 0^+$

$$b_j = \mathrm{Re}\, Q_j(e^{i\xi(\mu_c)}) \tag{4}$$

where the phase $e^{i\xi} \equiv {\epsilon'_k}^* / \epsilon'_k$ is defined in terms of the $\gamma \to 0^+$ limit of the derivative of the dielectric function ϵ_k. The identification of $e^{i\xi}$ as a scaling variable for the $\gamma \to 0^+$ regime which captures any remaining dependence on the underlying equilibrium is a novel result of this work.

The scaling $\rho(t) \sim \gamma^2$ for the amplitude of the unstable mode implies that the wave electric field follows the so-called trapping scaling $E \sim \gamma^2$ in the limit $\gamma \to 0^+$. The terminology arises from the equivalent scaling $\omega_b \sim \gamma$ between the bounce frequency of trapped

particles and the growth rate ($\omega_b^2 \equiv ekE_k/m$). Trapping scaling has been a characteristic feature of previous numerical simulations [8,9] as well as recent experiments [19]; however the theoretical results on the scaling of the saturated electric field are divided between theories predicting "Hopf scaling" $E \sim \sqrt{\gamma}$ [1,7,10,11], and theories predicting trapping scaling. [2–6,20] Perhaps the best known analysis leading to a prediction of Hopf scaling is a controversal paper by Simon and Rosenbluth. [7] Their work treats a one mode bump on tail instability by perturbatively expanding the Vlasov equation and seeking a time-periodic nonlinear solution. The perturbation theory leads to singular results and the final expressions are rendered finite by prescribing a regularization procedure. Subsequent perturbation theories have encountered comparable difficulties and proposed similar prescriptions. [10,11]

The approach I develop in this work differs crucially from these investigations in the interpretation and treatment of the singular behavior of the expansion at $\gamma = 0$. The derivation of the amplitude dynamics on the unstable manifold gives nonlinear coefficients as integrals over velocity, and as $\gamma \to 0^+$ these integrals diverge due to pinching singularities that develop at the phase velocity of the mode. As already mentioned, these divergences can be absorbed by simply rescaling the wave amplitude and this rescaling reveals an electric field that follows the trapping scaling. If, on the other hand, one were to regularize the integrals by somehow discarding the divergent part then the resulting equations for the mode amplitude would indeed scale as in dissipative Hopf bifurcation. This, in essence, is the step taken in the theories that predict Hopf scaling for the electric field.

The theory of invariant manifolds for equilibria of infinite-dimensional dynamical systems, such as partial differential equations, has been developed extensively in recent years with rigorous results establishing the existence and properties of these structures for various classes of evolution equations. Examples utilizing a variety of techniques are found in [21–26], and there is an introductory review by Vanderbauwhede and Iooss. [27] However, this theory does not yet treat equations such as the Vlasov-Poisson system and it seems to be an open problem to rigorously construct invariant manifolds for Vlasov equilibria. In this paper, unstable manifolds serve a heuristic role by motivating certain procedures for

constructing the expansions in the mode amplitude. Although these same expansions could certainly be set up without mentioning the manifolds, the dynamical systems viewpoint does seem to bring additional insight. Future development of a rigorous invariant manifold theory for the Vlasov equation may provide mathematical justification for these expansions. The present analysis is solely concerned with understanding the properties of the instability as represented by the amplitude expansions.

The remainder of the Introduction is devoted to defining our notation and summarizing relevant well known facts concerning the spectrum and eigenfunctions of the Vlasov-Poisson equation. In Section II the description of the unstable manifold is briefly reviewed and the equations necessary to obtain the dynamics on the manifold are derived. These equations are solved using power series in the amplitude of the unstable mode in Section III and a detailed analysis of the lowest order term in this expansion reveals the divergence mentioned above. This divergence is shown to imply the trapping scaling for the mode amplitude. In Sections IV-VI, the structure of the expansions is examined to all orders. The increasing strength of the divergences is calculated and a detailed analysis is made of how the dynamics on the unstable manifold depends on the critical equilibrium $F_0(v,\mu_c)$ in the $\gamma\to 0^+$ limit. The mode amplitude dynamics, the electric field, and to a large extent the distribution function depend on $F_0(v,\mu_c)$ only through the derivative of the dielectric function ϵ_k'. This conclusion indicates a degree of universality to the dynamics of a weakly unstable electrostatic mode that has not been previously appreciated.

A. Notation

For a neutral collisionless plasma with a fixed ion density n_0, the electron distribution function $F(x,v,t)$ and the electrostatic potential $\Phi(x,t)$ satisfy the dimensionless Vlasov-Poisson equations (in one dimension)

$$\frac{\partial F}{\partial t}+v\frac{\partial F}{\partial x}+\frac{\partial \Phi}{\partial x}\frac{\partial F}{\partial v}=0 \tag{5}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \int_{-\infty}^{\infty} dv\, F(x,v,t) - 1. \tag{6}$$

The plasma length is L and periodic boundary conditions are assumed with normalization

$$\int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv\, F(x,v,t) = L. \tag{7}$$

Here x, t and v are measured in units of u/ω_p, ω_p^{-1} and u, respectively, where u is a chosen velocity scale and $\omega_p^2 = 4\pi e^2 n_0/m$. The electron charge and mass are $-e$ and m and the ions are singly charged. The dimensionless distribution function and potential are measured in units of u^{-1} and mu^2/e respectively.

An inner product between two functions $G_1(x,v)$ and $G_2(x,v)$ is defined by

$$(G_1, G_2) \equiv \int_{-L/2}^{L/2} dx\, < G_1, G_2 > \tag{8}$$

where $< G_1, G_2 >$ denotes the integration over v alone:

$$< G_1, G_2 > \equiv \int_{-\infty}^{\infty} dv\, G_1(x,v)^* G_2(x,v). \tag{9}$$

With periodic boundary conditions the allowed wavevectors are multiples of $k_c = 2\pi/L$, and the Fourier expansion of $G(x,v)$ will be written as

$$G(x,v) = \sum_{k=-\infty}^{\infty} e^{ikk_c x}\, G_k(v). \tag{10}$$

Let $F_0(v,\mu)$ denote a parametrized family of equilibria, normalized to unit density

$$\int_{-\infty}^{\infty} dv\, F_0(v,\mu) = 1, \tag{11}$$

and re-express the distribution function relative to this family $F(x,v,t) = F_0(v,\mu) + f(x,v,t)$, then f satisfies

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{\partial \Phi}{\partial x}\frac{\partial F_0}{\partial v} + \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v} = 0 \tag{12}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \int_{-\infty}^{\infty} dv\, f(x,v,t) \tag{13}$$

and

$$\int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv\, f(x,v,t) = 0. \tag{14}$$

With the Fourier series for f, equations (12) - (13) can be combined

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + \mathcal{N}(f) \tag{15}$$

where the linear operator is defined by

$$\mathcal{L}\, f = \sum_{k=-\infty}^{\infty} e^{ikk_c x}\, (L_k f_k)(v) \tag{16}$$

$$(L_k f_k)(v) = \begin{cases} 0 & k = 0 \\ -ikk_c\left[v f_k(v) + k^{-2}\, \eta(v,\mu) \int_{-\infty}^{\infty} dv'\, f_k(v')\right] & k \neq 0 \end{cases} \tag{17}$$

with

$$\eta(v,\mu) = -k_c^{-2} \frac{\partial F_0}{\partial v}(v,\mu), \tag{18}$$

and

$$\mathcal{N}(f) = \frac{i}{k_c} \sum_{k=-\infty}^{\infty} e^{ikk_c x} \sum_{l=-\infty}^{\infty}{}' \frac{1}{l} \frac{\partial f_{k-l}}{\partial v} \int_{-\infty}^{\infty} dv'\, f_l(v'). \tag{19}$$

Here and below a primed summation omits the $l = 0$ term.

Symmetries of the model (5) - (6) and the equilibrium $F_0(v,\mu)$ are important qualitative features of the problem. Spatial translation, $\mathcal{T}_a : (x,v) \rightarrow (x+a, v)$, and reflection, $\kappa : (x,v) \rightarrow (-x,-v)$, act as operators on the distribution function in the usual way: if α denotes an arbitrary transformation then $(\alpha \cdot f)(x,v) \equiv f(\alpha^{-1} \cdot (x,v))$. The operators $\mathcal{L}$ and $\mathcal{N}$ commute with $\mathcal{T}_a$ due to the spatial homogeneity of F_0, and if $F_0(v,\mu) = F_0(-v,\mu)$, then $\mathcal{L}$ and $\mathcal{N}$ also commute with the reflection operator κ. Together $\mathcal{T}_a$ and κ generate the symmetry group of the circle O(2) and without κ the symmetry drops to SO(2).

The spectral theory for $\mathcal{L}$ is well established, and the needed results are simply recalled to establish the notation. [17,28,29] The eigenvalues $\lambda = -ikk_c z$ of $\mathcal{L}$ are determined by the roots $\Lambda_k(z,\mu) = 0$ of the "spectral function",

$$\Lambda_k(z,\mu) \equiv 1 + \frac{1}{k^2}\int_{-\infty}^{\infty} dv\,\frac{\eta(v,\mu)}{v-z}. \tag{20}$$

Unless it is necessary to manipulate the parameter dependence of $\Lambda_k(z,\mu)$, the argument μ will be suppressed. If only translation symmetry is present, these eigenvalues are generically complex; when the equilibrium is also reflection-symmetric then real eigenvalues can arise, for example in a two-stream instability. [30]

The spectral function is analytic in the upper and lower half planes with a branch cut on the real axis along the support of η. For $z = r \pm i\epsilon$, the discontinuity across the cut is given by the Plemelj formula [31]

$$\lim_{\epsilon\to 0^+} \Lambda_k(r \pm i\epsilon) = 1 + \frac{1}{k^2}\left[\text{P.V.}\int_{-\infty}^{\infty} dv\,\frac{\eta(v,\mu)}{v-r}\right] \pm i\pi\eta(r,\mu). \tag{21}$$

The analytic continuation of $\Lambda_k(z)$ from $\text{Im}\, z > 0$ to $\text{Im}\, z < 0$ yields the dielectric function $\epsilon_k(z)$ [32], defined in the usual way via the Landau contour. [33] Since our analysis focuses on the regime $\text{Im}\, z \geq 0$, the notations $\Lambda_k(z)$ and $\epsilon_k(z)$ are interchangeable in the subsequent discussion.

The branch cut of $\Lambda_k(z)$ corresponds to a continuous spectrum for $\mathcal{L}$ on the imaginary axis. As μ varies the roots of $\Lambda_k(z)$ typically vary; in particular, roots can appear or disappear through the branch cut. The appearance of a root at the cut corresponds to the birth of eigenvalues embedded in the continuous spectrum and this occurs for the critical parameter values μ_c marking the threshold of linear instability. From (21) such a real root r must satisfy

$$\eta(r,\mu_c) = 0 \tag{22}$$

$$1 + \frac{1}{k^2}\left[\text{P.V.}\int_{-\infty}^{\infty} dv\,\frac{\eta(v,\mu_c)}{v-r}\right] = 0. \tag{23}$$

Corresponding to the eigenvalue $\lambda = -ikk_c z$ is an eigenfunction

$$\Psi(x,v) = e^{ikk_c x}\,\psi(v) \tag{24}$$

with

$$\psi(v) = \left(-\frac{1}{k^2}\right)\frac{\eta(v,\mu)}{v-z}. \tag{25}$$

There is also an adjoint eigenfunction $\tilde{\Psi}(x,v)$ satisfying $(\tilde{\Psi},\Psi) = 1$ given by

$$\tilde{\Psi}(x,v) = \frac{1}{L}e^{ikk_c x}\tilde{\psi}(v) \tag{26}$$

where

$$\tilde{\psi}(v) \equiv -\left(\frac{1}{\Lambda_k'(z)(v-z)}\right)^*. \tag{27}$$

The normalization in (27) assumes that the root of $\Lambda_k(z)$ is simple and is chosen so that $<\tilde{\psi},\psi>= 1$.

II. AMPLITUDE EQUATION ON THE UNSTABLE MANIFOLD

Since Landau damping is weakest at long wavelengths, one expects that instability will occur first at $k_c = 2\pi/L$ as μ is varied through μ_c. This point has recently been treated pedagogically by Shadwick and Morrison. [34] The critical eigenvalue is then $\lambda = -ik_c z_0$ corresponding to $\Lambda_1(z_0) = 0$ for $k = 1$. The root z_0 determines the phase velocity v_p and the growth rate γ of the linear mode

$$z_0 = \frac{i\gamma}{k_c} + v_p; \tag{28}$$

both v_p and γ depend on μ. However it is more convenient to take γ as the independent parameter and regard $\mu(\gamma)$, $z_0(\gamma)$, and $v_p(\gamma)$ as functions of the growth rate. Thus $\Lambda_1(z_0) = 0$ is understood to mean

$$\Lambda_1(z_0(\gamma),\mu(\gamma)) = 1 + \int_{-\infty}^{\infty}\frac{dv\,\eta(v,\mu(\gamma))}{v - v_p(\gamma) - i\gamma/k_c} = 0. \tag{29}$$

In Appendix A, this equation is solved for $v_p(\gamma)$ and $\mu(\gamma)$ to first order in γ. From (28), $\lambda = -ik_c z_0(\gamma)$ becomes $\lambda = \gamma - i\omega(\gamma)$ where $\omega(\gamma) = k_c v_p(\gamma)$. The notation $\gamma \to 0^+$ for the weak growth rate regime always refers to the joint limit

$$(\gamma,\omega(\gamma),\mu(\gamma)) \to (0,\omega(0),\mu(0)) \equiv (0,\omega_c,\mu_c). \tag{30}$$

Subsequently, the γ argument for v_p, ω, z_0, and μ will generally be suppressed when it is not explicitly required.

I assume that z_0 is a simple root:

$$\Lambda_1'(z_0) \equiv \frac{d\,\Lambda_1}{dz}(z_0) \neq 0; \tag{31}$$

in addition, at $z = z_0$ all derivatives of $\Lambda_k(z)$ are assumed to have finite limits: $\lim_{\gamma\to 0^+} |\Lambda_k^{(j)}(z_0)| < \infty$ where

$$\Lambda_k^{(j)}(z) \equiv \frac{d^j\,\Lambda_k}{dz^j}(z) = \frac{j!}{k^2}\int_{-\infty}^{\infty} dv\, \frac{\eta(v,\mu)}{(v-z)^{j+1}}. \tag{32}$$

From (20) and (29) $\Lambda_k(z_0)$ can be evaluated for arbitrary k

$$\Lambda_k(z_0) = \frac{k^2-1}{k^2}; \tag{33}$$

this identity will be needed below. Since $\Lambda_k(z)$ and $\epsilon_k(z)$ are identical for Im $z \geq 0$, the relations (31) - (33) are unchanged if $\Lambda_k(z_0)$ is replaced by $\epsilon_k(z_0)$.

The unstable mode corresponding to z_0 is

$$\Psi_1(x,v) = e^{ik_c x}\,\psi_c(v) \equiv e^{ik_c x}\left(\frac{-\eta(v,\mu)}{v-z_0}\right). \tag{34}$$

When $F_0(v,\mu)$ lacks reflection symmetry, then this wave typically has a non-zero phase velocity and λ is complex. In this case the identities $\Lambda_k(z) = \Lambda_{-k}(z)$ and $\Lambda_k(z)^* = \Lambda_k(z^*)$ imply three additional modes: Ψ_1^*, Ψ_2, and Ψ_2^* where

$$\Psi_2(x,v) = e^{ik_c x}\left(\frac{-\eta(v,\mu)}{v-z_0^*}\right). \tag{35}$$

These eigenfunctions correspond to eigenvalues λ^*, $-\lambda^*$, and $-\lambda$, respectively, and fill out the eigenvalue quartet characteristic of Hamiltonian systems.

In the event that $F_0(v,\mu)$ is reflection-symmetric in v, then both real and complex eigenvalues may occur. If λ is complex, then since κ and $\mathcal{L}$ commute, Ψ_1 and

$$(\kappa\cdot\Psi_1)(x,v) = e^{-ik_c x}\,\psi_c(-v) \tag{36}$$

are linearly independent eigenvectors for the same eigenvalue $\lambda = -ik_c z_0$. Thus λ generically has multiplicity two. The same considerations hold for Ψ_1^*, Ψ_2, and Ψ_2^* so the entire quartet has double mutiplicity. When λ is real as in the symmetric two-stream instability, then one can show that $\kappa \cdot \Psi_1 = \Psi_1^*$ and λ is again multiplicity two. The eigenvectors Ψ_2 and $\kappa \cdot \Psi_2 = \Psi_2^*$ correspond to $-\lambda$.

A. Prototypical example

A convenient and explicit family of equilibria, satisfying (11), is

$$F_0(v,\mu) = \frac{1}{\pi}\left[\frac{n}{(v-u_p)^2+1} + \frac{\Delta(1-n)}{(v-u_b)^2+\Delta^2}\right] \tag{37}$$

with parameter set $\mu = (n, u_p, u_b, \Delta)$. In this example, one component, the plasma, has density $n\, n_0$ and the second component, the beam, has density $(1-n)n_0$; each component has its own drift velocity and the beam has thermal width Δ. The thermal width of the plasma has been taken as the velocity unit. If $n = 0.5$, $\Delta = 1$ and $u_p = -u_b$, then the family has reflection symmetry.

In the four-dimensional parameter space the threshold of linear instability corresponds to a three-dimensional surface which is denoted by μ_c. The equilibria $F_0(v, \mu_c)$ on this surface have eigenvalues embedded in the continuous spectrum.

An illustrative realization of a one mode beam-plasma instability is shown in Fig. 1 for a system with $L = 2\pi$, $n = 0.8$, $u_p = 0.0$, $\Delta = 0.3$, and u_b varied to produce the instability of the $k = 1$ mode. The spectrum of $\mathcal{L}$ for the stable (a), critical (b) and unstable (c) regimes is illustrated in Fig. 2.

B. Critical linear modes

In this paper only the simplest instabilities having two unstable eigenvectors are considered. This setting nevertheless encompasses both the case of a reflection-symmetric instability with a real eigenvalue (two-stream) and the case of a complex conjugate eigenvalue

pair without reflection symmetry (beam-plasma). In either case, the components of the distribution function along the critical eigenvectors Ψ_1 and Ψ_1^* are separated out by writing

$$f(x,v,t) = [A(t)\Psi(x,v) + cc] + S(x,v,t) \tag{38}$$

where $A(t) = (\tilde{\Psi}, f)$ is the mode amplitude for Ψ and $(\tilde{\Psi}, S) = 0$. In (38) the subscript on Ψ_1 has been dropped, and $\tilde{\Psi} = \exp(ik_c x)\,\tilde{\psi}_c/L$ is the adjoint function for z_0 given in (26). The action of the translations $\mathcal{T}_a$ and reflection κ on the distribution function implies an action by these operators on the mode amplitudes: from (38) we have

$$\mathcal{T}_a \cdot A = e^{-ik_c a} A \tag{39}$$

$$\kappa \cdot A = A^*. \tag{40}$$

When these transformations are symmetries, these relations are useful for organizing the amplitude expansions below.

The Vlasov equation (15) determines the dynamics for A and S:

$$\dot{A} = \lambda\, A + (\tilde{\Psi}, \mathcal{N}(f)) \tag{41}$$

$$\frac{\partial S}{\partial t} = \mathcal{L}S + \mathcal{N}(f) - \Big[(\tilde{\Psi}, \mathcal{N}(f))\,\Psi + cc\Big] \tag{42}$$

where

$$(\tilde{\Psi}, \mathcal{N}(f)) = -\frac{i}{k_c} \sum_{l=-\infty}^{\infty}{}' \frac{1}{l} < \partial_v\, \tilde{\psi}_c, f_{1-l} > \int_{-\infty}^{\infty} dv'\, f_l(v'). \tag{43}$$

In writing (41) I have used the adjoint relationship $(\tilde{\Psi}, \mathcal{L}S) = (\mathcal{L}^\dagger \tilde{\Psi}, S) = \lambda^*(\tilde{\Psi}, S) = 0$ and in (43) an integration by parts $< \tilde{\psi}_c, \partial_v\, f_{k-l} >= - < \partial_v\, \tilde{\psi}_c, f_{k-l} >$ moves the velocity derivative in (19) onto $\tilde{\psi}_c$.

C. Amplitude equation on the unstable manifold

In (41) and (42) the critical modes are linearly decoupled from the other degrees of freedom but remain coupled to S through the nonlinear terms. For $\gamma > 0$, by restricting

to the dynamics on the unstable manifold, one can decouple the nonlinear terms as well and obtain from (41) an autonomous description of the dynamics of A as a two-dimensional flow. This reduction to a two-dimensional submanifold is analogous to the familiar procedure of center manifold reduction in dissipative bifurcation theory; here the unstable manifold partially compensates for the absence of a low-dimensional center manifold at criticality. [12,13]

The essential properties of an unstable manifold W^u are briefly described, more detail can be found in the extensive dynamical systems literature. [12,13] The unstable modes Ψ and Ψ^* span a two-dimensional unstable subspace E^u and the remaining spectrum of $\mathcal{L}$ determines the center subspace E^c (for spectrum on the imaginary axis) and a two-dimensional stable subspace E^s, spanned by the two stable modes; see Fig. 2(c). These subspaces are invariant under the linear flow [35], $\partial_t f = \mathcal{L} f$, but this invariance is lost when the nonlinear terms $\mathcal{N}(f)$ couple E^u to $E^c \oplus E^s$. However there are nonlinear manifolds present for the full nonlinear flow that are analogous to the subspaces of linear theory. Specifically, the unstable manifold W^u is invariant under the nonlinear evolution, $\partial_t f = \mathcal{L} f + \mathcal{N}(f)$, and tangent to E^u at the equilibrium $F_0(v, \mu)$; hence W^u is also two-dimensional. Solutions on this manifold $f^u(x, v, t)$ asymptotically approach F_0 as $t \to -\infty$.

The invariance of the unstable manifold means that restricting the Vlasov equation to W^u yields an autonomous two-dimensional dynamical system describing the evolution of initial conditions on the manifold, $f^u(x, v, 0) \in W^u$. Near the equilibrium this restriction is tractable since the tangency between W^u and the unstable subspace allows the manifold to be described by a function,

$$H : E^u \to E^c \oplus E^s \tag{44}$$

$$(A, A^*) \to H(x, v, A, A^*) \tag{45}$$

which measures the "distance" from E^u to W^u; see Fig. 3. Thus near F_0, for trajectories $f^u(x, v, t)$ on W^u, the evolution of the non-critical modes $S(x, v, t)$ in (38) is controlled by the critical modes:

$$S(x,v,t) = H(x,v,A(t),A^*(t)), \tag{46}$$

and these trajectories can be described entirely in terms of H and the evolution of $A(t)$:

$$f^u(x,v,t) = [A(t)\Psi(x,v) + cc] + H(x,v,A(t),A^*(t)). \tag{47}$$

If H is known, then using (47), the general equations (41) and (42) can be restricted to the unstable manifold:

$$\dot{A} = \lambda\, A + (\tilde{\Psi}, \mathcal{N}(f^u)) \tag{48}$$

$$\left.\frac{\partial S}{\partial t}\right|_{f^u} = \mathcal{L}H + \mathcal{N}(f^u) - \left[(\tilde{\Psi}, \mathcal{N}(f^u))\,\Psi + cc\right]. \tag{49}$$

Now (48) defines an *autonomous* two-dimensional flow describing the self-consistent nonlinear evolution of the unstable mode; this is the amplitude equation I wish to study.

Translation symmetry forces the right hand side of (48) to have the form

$$\lambda\, A + (\tilde{\Psi}, \mathcal{N}(f^u)) = A\, p(\sigma, \mu) \tag{50}$$

where $\sigma \equiv |A|^2$ and the function $p(\bullet, \mu)$ is not constrained by the translation symmetry. [36] Typically $p(\sigma,\mu)$ is complex-valued, however when F_0 is reflection-symmetric then $p(\sigma,\mu)$ is forced to be real.

It is convenient to view (50) as expressing p in terms of H, and make this connection more explicit by evaluating $(\tilde{\Psi}, \mathcal{N}(f^u))$. This calculation requires the Fourier series for H

$$H(x,v,A,A^*) = \sum_{k=-\infty}^{\infty} e^{ikk_cx}\, H_k(v,A,A^*), \tag{51}$$

and the Fourier components of f^u

$$f^u_k(v) = [A\psi_c(v)\,\delta_{k,1} + A^*\psi_c(v)^*\,\delta_{k,-1}] + H_k(v,A,A^*). \tag{52}$$

The evaluation of $(\tilde{\Psi}, \mathcal{N}(f^u))$ from (43) is simplified by first noting that the components of H are forced by translation symmetry to have the form

$$\begin{aligned} H_0(v,A,A^*) &= \sigma\, h_0(v,\sigma) \\ H_1(v,A,A^*) &= A\sigma\, h_1(v,\sigma) \\ H_k(v,A,A^*) &= A^k\, h_k(v,\sigma) \quad \text{for } \; k \geq 2, \end{aligned} \tag{53}$$

where $H_{-k} = H_k^*$ and the functions h_k are not constrained by the translations. [37] However, if reflection symmetry also holds, then

$$h_k(-v,\sigma) = h_k(v,\sigma)^*. \tag{54}$$

Combining (52) - (53) with $(\tilde{\Psi}, \mathcal{N}(f^u))$ in (43) yields

$$\begin{aligned}(\tilde{\Psi}, \mathcal{N}(f^u)) = \frac{-iA\sigma}{k_c} \Bigg\{ &< \partial_v \tilde{\psi}_c, (h_0 - h_2) > + \frac{\Gamma_2}{2} < \partial_v \tilde{\psi}_c, \psi_c^* > \\ &+\sigma \left[\Gamma_1 < \partial_v \tilde{\psi}_c, h_0 > - \Gamma_1^* < \partial_v \tilde{\psi}_c, h_2 > + \frac{\Gamma_2}{2} < \partial_v \tilde{\psi}_c, h_1^* > - \frac{\Gamma_2^*}{2} < \partial_v \tilde{\psi}_c, h_3 > \right] \\ &+ \sum_{l=3}^{\infty} \frac{\sigma^{l-2}}{l} [\Gamma_l < \partial_v \tilde{\psi}_c, h_{l-1}^* > - \sigma \Gamma_l^* < \partial_v \tilde{\psi}_c, h_{l+1} >] \Bigg\}\end{aligned} \tag{55}$$

where, on the right hand side, the velocity integral of h_k is denoted by

$$\Gamma_k(\sigma) \equiv \int_{-\infty}^{\infty} dv' \, h_k(v', \sigma). \tag{56}$$

Now comparing (50) and (55) provides the desired expression for p

$$\begin{aligned}p(\sigma,\mu) = \lambda - \frac{i\sigma}{k_c} \Bigg\{ &< \partial_v \tilde{\psi}_c, (h_0 - h_2) > + \frac{\Gamma_2}{2} < \partial_v \tilde{\psi}_c, \psi_c^* > \\ &+\sigma \left[\Gamma_1 < \partial_v \tilde{\psi}_c, h_0 > - \Gamma_1^* < \partial_v \tilde{\psi}_c, h_2 > + \frac{\Gamma_2}{2} < \partial_v \tilde{\psi}_c, h_1^* > - \frac{\Gamma_2^*}{2} < \partial_v \tilde{\psi}_c, h_3 > \right] \\ &+ \sum_{l=3}^{\infty} \frac{\sigma^{l-2}}{l} [\Gamma_l < \partial_v \tilde{\psi}_c, h_{l-1}^* > - \sigma \Gamma_l^* < \partial_v \tilde{\psi}_c, h_{l+1} >] \Bigg\}.\end{aligned} \tag{57}$$

In order to exploit this expression for p, it is necessary to determine H or equivalently to determine the functions h_k.

D. Representation of the unstable manifold

An equation for H follows by requiring consistency between (46) and (49); setting the time derivative of (46) equal to the right hand side of (49) gives

$$\frac{\partial H}{\partial A} \dot{A} + \frac{\partial H}{\partial A^*} \dot{A}^* = \mathcal{L} \, H + \mathcal{N}(f^u) - \left[(\tilde{\Psi}, \mathcal{N}(f^u)) \, \Psi + cc \right] \tag{58}$$

which is to be solved for H subject to $H(x,v,0,0) = 0$ and

$$\frac{\partial H}{\partial A}(x,v,0,0) = \frac{\partial H}{\partial A^*}(x,v,0,0) = 0. \tag{59}$$

These latter conditions are implied by the tangency between W^u and E^u at the equilibrium $(A, A^*) = (0,0)$, and are automatically satisfied in this case by virtue of (53).

With the previous expression for $(\tilde{\Psi}, \mathcal{N}(f^u))$ in (55) and the notation in (53) for the Fourier components of H, the components of (58) take the form

$$\begin{aligned}
&\frac{\partial H_0}{\partial A}\,\dot{A} + \frac{\partial H_0}{\partial A^*}\,\dot{A}^* = \\
&\qquad \frac{i\sigma}{k_c}\,\frac{\partial}{\partial v}\left\{\left[\psi_c^* + \sigma(h_1^* - \psi_c\Gamma_1^*) + \sigma^2\, h_1^*\Gamma_1 + \sum_{l=2}^{\infty}\frac{\sigma^{l-1}}{l} h_l^*\Gamma_l\right] - cc\right\}
\end{aligned} \tag{60}$$

$$\begin{aligned}
&\frac{\partial H_1}{\partial A}\,\dot{A} + \frac{\partial H_1}{\partial A^*}\,\dot{A}^* - L_1\,H_1 = \\
&\qquad \frac{iA\sigma}{k_c}\,\mathcal{P}_\perp\frac{\partial}{\partial v}\left\{h_0 - h_2 + \frac{1}{2}\psi_c^*\,\Gamma_2 + \sigma\left[h_0\Gamma_1 - h_2\Gamma_1^* + \frac{1}{2}h_1^*\Gamma_2 - \frac{1}{2}h_3\Gamma_2^*\right]\right. \\
&\qquad\qquad \left. + \sum_{l=3}^{\infty}\frac{\sigma^{l-2}}{l}\left[h_{l-1}^*\Gamma_l - \sigma h_{l+1}\Gamma_l^*\right]\right\}
\end{aligned} \tag{61}$$

$$\begin{aligned}
&\frac{\partial H_2}{\partial A}\,\dot{A} + \frac{\partial H_2}{\partial A^*}\,\dot{A}^* - L_2\,H_2 = \\
&\qquad \frac{iA^2}{k_c}\,\frac{\partial}{\partial v}\left\{\psi_c + \sigma\left[h_1 + \psi_c\Gamma_1 - h_3 + \frac{1}{2}h_0\Gamma_2 + \frac{1}{3}\psi_c^*\Gamma_3\right]\right. \\
&\qquad\qquad \sigma^2\left[h_1\Gamma_1 - h_3\Gamma_1^* - \frac{1}{2}h_4\Gamma_2^* + \frac{1}{3}h_1^*\Gamma_3\right] \\
&\qquad\qquad \left. -\frac{\sigma^3}{3}h_5\Gamma_3^* + \sum_{l=4}^{\infty}\frac{\sigma^{l-2}}{l}\left[h_{l-2}^*\Gamma_l - \sigma^2 h_{l+2}\Gamma_l^*\right]\right\}
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
&\frac{\partial H_k}{\partial A}\,\dot{A} + \frac{\partial H_k}{\partial A^*}\,\dot{A}^* - L_k\,H_k = \\
&\qquad \frac{iA^k}{k_c}\,\frac{\partial}{\partial v}\left\{h_{k-1} + \frac{\psi_c}{k-1}\Gamma_{k-1} + \sum_{l=2}^{k-2}\frac{h_{k-l}}{l}\Gamma_l\right. \\
&\qquad\qquad +\sigma\left[h_{k-1}\Gamma_1 - h_{k+1} + \frac{h_1}{k-1}\Gamma_{k-1} + \frac{h_0}{k}\Gamma_k + \frac{\psi_c^*}{k+1}\Gamma_{k+1}\right] \\
&\qquad\qquad +\sigma^2\left[-h_{k+1}\Gamma_1^* + \frac{h_1^*}{k+1}\Gamma_{k+1}\right] + \sum_{l=k+2}^{\infty}\frac{\sigma^{l-k}}{l}h_{l-k}^*\Gamma_l \\
&\qquad\qquad \left. -\sum_{l=2}^{\infty}\frac{\sigma^l}{l}h_{k+l}\Gamma_l^*\right\}
\end{aligned} \tag{63}$$

for $k = 0, 1, 2$, and $k > 2$, respectively. In the $k = 1$ component (61), $\mathcal{P}$ is the projection operator onto the $\psi_c(v)$ component of a function $g(v)$,

$$(\mathcal{P}g)(v) \equiv < \tilde{\psi}_c, g > \psi_c(v), \tag{64}$$

and the orthogonal projection is denoted by $\mathcal{P}_\perp \equiv I - \mathcal{P}$.

The expressions for $\dot{A}\partial_A H_k + \dot{A}^* \partial_{A^*} H_k$ can also be evaluated in terms of the functions $h_k(v, \sigma)$ and $p(\sigma, \mu)$:

$$\frac{\partial H_0}{\partial A}\,\dot{A} + \frac{\partial H_0}{\partial A^*}\,\dot{A}^* = \sigma(p + p^*)\left[h_0 + \sigma\frac{\partial h_0}{\partial \sigma}\right] \tag{65}$$

$$\frac{\partial H_1}{\partial A}\,\dot{A} + \frac{\partial H_1}{\partial A^*}\,\dot{A}^* - L_1\,H_1 = A\sigma\left[\{(2p + p^*) - L_1\}h_1 + (p + p^*)\sigma\frac{\partial h_1}{\partial \sigma}\right] \tag{66}$$

and

$$\frac{\partial H_k}{\partial A}\,\dot{A} + \frac{\partial H_k}{\partial A^*}\,\dot{A}^* - L_k\,H_k = A^k\left[\{kp - L_k\}h_k + (p + p^*)\sigma\frac{\partial h_k}{\partial \sigma}\right] \tag{67}$$

for $k = 0, 1$, and $k \geq 2$, respectively. By combining (60) - (63) and (65) - (67), the component equations of (58) reduce to a simpler form:

$$\begin{aligned}(p + p^*)\left[h_0 + \sigma\frac{\partial h_0}{\partial \sigma}\right] &= \\ &\frac{i}{k_c}\frac{\partial}{\partial v}\left\{\left[\psi_c^* + \sigma(h_1^* - \psi_c\Gamma_1^*) + \sigma^2\,h_1^*\Gamma_1 + \sum_{l=2}^{\infty}\frac{\sigma^{l-1}}{l}h_l^*\Gamma_l\right] - cc\right\}\end{aligned} \tag{68}$$

$$\begin{aligned}\left[\{(2p + p^*) - L_1\}h_1 + (p + p^*)\sigma\frac{\partial h_1}{\partial \sigma}\right] &= \\ &\frac{i}{k_c}\mathcal{P}_\perp\frac{\partial}{\partial v}\left\{h_0 - h_2 + \frac{1}{2}\psi_c^*\,\Gamma_2 + \sigma\left[h_0\Gamma_1 - h_2\Gamma_1^* + \frac{1}{2}h_1^*\Gamma_2 - \frac{1}{2}h_3\Gamma_2^*\right]\right. \\ &\left. + \sum_{l=3}^{\infty}\frac{\sigma^{l-2}}{l}\left[h_{l-1}^*\Gamma_l - \sigma h_{l+1}\Gamma_l^*\right]\right\}\end{aligned} \tag{69}$$

$$\left[\{2p - L_2\}h_2 + (p + p^*)\sigma\frac{\partial h_2}{\partial \sigma}\right] = \tag{70}$$

$$\frac{i}{k_c}\frac{\partial}{\partial v}\left\{\psi_c + \sigma\left[h_1 + \psi_c\Gamma_1 - h_3 + \frac{1}{2}h_0\Gamma_2 + \frac{1}{3}\psi_c^*\Gamma_3\right]\right.$$
$$\sigma^2\left[h_1\Gamma_1 - h_3\Gamma_1^* - \frac{1}{2}h_4\Gamma_2^* + \frac{1}{3}h_1^*\Gamma_3\right]$$
$$\left. -\frac{\sigma^3}{3}h_5\Gamma_3^* + \sum_{l=4}^{\infty}\frac{\sigma^{l-2}}{l}\left[h_{l-2}^*\Gamma_l - \sigma^2 h_{l+2}\Gamma_l^*\right]\right\}$$

and

$$\left[\{kp - L_k\}h_k + (p+p^*)\sigma\frac{\partial h_k}{\partial\sigma}\right] = \tag{71}$$
$$\frac{i}{k_c}\frac{\partial}{\partial v}\left\{h_{k-1} + \frac{\psi_c}{k-1}\Gamma_{k-1} + \sum_{l=2}^{k-2}\frac{h_{k-l}}{l}\Gamma_l\right.$$
$$+\sigma\left[h_{k-1}\Gamma_1 - h_{k+1} + \frac{h_1}{k-1}\Gamma_{k-1} + \frac{h_0}{k}\Gamma_k + \frac{\psi_c^*}{k+1}\Gamma_{k+1}\right]$$
$$+\sigma^2\left[-h_{k+1}\Gamma_1^* + \frac{h_1^*}{k+1}\Gamma_{k+1}\right] + \sum_{l=k+2}^{\infty}\frac{\sigma^{l-k}}{l}h_{l-k}^*\Gamma_l$$
$$\left. -\sum_{l=2}^{\infty}\frac{\sigma^l}{l}h_{k+l}\Gamma_l^*\right\}$$

for $k = 0, 1, 2$, and $k > 2$, respectively.

Together with (57) these component equations determine the functions $p(\sigma, \mu)$ and $\{h_k(\sigma,\mu)\}_{k=0}^{\infty}$. From a practical point of view, all that has been achieved to this point is a reduction of the problem to the analysis of functions of a *single* real variable, i.e. σ. However, in the study of the amplitude equation (48), this reduction does provide a useful simplification which is exploited in the discussion below.

III. EXPANSIONS, RECURSION RELATIONS, AND PINCHING SINGULARITIES

The amplitude equation on the unstable manifold,

$$\dot{A} = A\, p(\sigma, \mu), \tag{72}$$

is analyzed by expressing p as a power series in the mode amplitude,

$$p(\sigma,\mu) = \sum_{j=0}^{\infty} p_j(\mu)\,\sigma^j, \tag{73}$$

whose coefficients $p_j(\mu)$ are calculated from (57) using an analogous series for h_k

$$h_k(v,\sigma) = \sum_{j=0}^{\infty} h_{k,j}(v)\,\sigma^j. \tag{74}$$

For notation, denote the integral over $h_{k,j}(v)$ by

$$\Gamma_{k,j} \equiv \int_{-\infty}^{\infty} dv\, h_{k,j}(v) \tag{75}$$

so that

$$\Gamma_k(\sigma) = \sum_{j=0}^{\infty} \Gamma_{k,j}\,\sigma^j \tag{76}$$

from (56), then expanding the expression for p in (57) gives the coefficients $p_j(\mu)$ as

$$p_0 = \lambda \tag{77}$$

$$p_j = \frac{i}{k_c}[\mathcal{A}_j + \mathcal{B}_j] \quad \text{for} \quad j \geq 1 \tag{78}$$

where

$$\begin{aligned} \mathcal{A}_j =& - < \partial_v \tilde{\psi}_c, (h_{0,j-1} - h_{2,j-1}) > -\frac{1}{2} < \partial_v \tilde{\psi}_c, \psi_c^* > \Gamma_{2,j-1} \\ & - \sum_{l=0}^{j-2} \left(< \partial_v \tilde{\psi}_c, h_{0,j-l-2} > \Gamma_{1,l} - < \partial_v \tilde{\psi}_c, h_{2,j-l-2} > \Gamma_{1,l}^* \right) \end{aligned} \tag{79}$$

$$\begin{aligned} \mathcal{B}_j =& - \sum_{l=0}^{j-2} \left\{ \frac{1}{2} < \partial_v \tilde{\psi}_c, h_{1,j-l-2}^* > \Gamma_{2,l} + \sum_{m=0}^{l} \left[\frac{\Gamma_{j-l+1,m}}{j-l+1} < \partial_v \tilde{\psi}_c, h_{j-l,l-m}^* > \right.\right. \\ & \left.\left. - \frac{\Gamma_{j-l,m}^*}{j-l} < \partial_v \tilde{\psi}_c, h_{j-l+1,l-m} > \right] \right\}. \end{aligned} \tag{80}$$

Here and below, a summation is understood to be omitted if the lower limit exceeds the upper limit. The organization of terms between (79) and (80) will turn out to distinguish different singular behaviors in the $\gamma \to 0^+$ limit: $\mathcal{A}_j \sim \gamma^{-(4j-1)}$ and $\mathcal{B}_j \sim \gamma^{-(4j-2)}$; thus the $\mathcal{B}_j$ terms are sub-dominant in the weak growth rate regime.

The leading term in (73) is the eigenvalue $p_0 = \lambda$, and higher order terms are determined by calculating $h_{k,j}(v)$ from (68) - (71); the results are summarized below. For $k \neq 0$ the

coefficients $h_{k,j}(v)$ are expressed in terms of the resolvent operator $R_k(w) \equiv (w - L_k)^{-1}$. For an arbitrary complex number w, the resolvent acts on a function $g(v)$ by [38]

$$(R_k(w)\,g)(v) = \frac{1}{ikk_c(v - iw/kk_c)}\left[g(v) - \frac{\eta(v,\mu)}{k^2\,\Lambda_k(iw/kk_c)}\int_{-\infty}^{\infty} dv'\,\frac{g(v')}{v' - iw/kk_c}\right]. \tag{81}$$

A. Series coefficients for $h_k(v,\sigma)$

For $k = 0$ the coefficients $h_{0,l}$ are found from (68); inserting the expansions for $h_k(v,\sigma)$ and $\Gamma_k(\sigma)$ into (68) and setting the coefficient of σ^l to zero yields

$$h_{0,l}(v) = \frac{I_{0,l}(v)}{(1+l)(\lambda+\lambda^*)}. \tag{82}$$

The functions $I_{0,l}(v)$ are given by

$$I_{0,0}(v) = \frac{i}{k_c}\frac{\partial}{\partial v}(\psi_c^* - \psi_c) \tag{83}$$

and for $l \geq 1$

$$\begin{aligned} I_{0,l}(v) = &-\sum_{j=0}^{l-1}(1+j)(p_{l-j} + p_{l-j}^*)\,h_{0,j}(v) \\ &+\frac{i}{k_c}\frac{\partial}{\partial v}\Bigg\{\Bigg[h_{1,l-1}^* - \psi_c\Gamma_{1,l-1}^* + \sum_{j=0}^{l-2} h_{1,j}^*\Gamma_{1,l-j-2} \\ &\qquad + \sum_{j=0}^{l-1}\sum_{j'=0}^{j}\frac{h_{l-j+1,j'}^*}{l-j+1}\Gamma_{l-j+1,j-j'}\Bigg] - cc\Bigg\}. \end{aligned} \tag{84}$$

For $k \geq 1$, following the same procedure in (69) - (71) determines the corresponding expressions for $h_{k,l}$; these coefficients have the form

$$h_{k,l}(v) = R_k(w_{k,l})\,I_{k,l} \tag{85}$$

where $R_k(w)$ is given in (81) and

$$w_{k,l} \equiv (l + \delta_{k,1})(\lambda + \lambda^*) + k\lambda = 2(l + \delta_{k,1})\,\gamma + k\lambda. \tag{86}$$

For $k = 1$, the functions $I_{1,l}(v)$ are given by

$$
\begin{aligned}
I_{1,l}(v) = &-\sum_{j=0}^{l-1}\left[(2+j)p_{l-j}+(1+j)p^*_{l-j}\right]h_{1,j} \\
&+\frac{i}{k_c}\mathcal{P}_\perp\frac{\partial}{\partial v}\Bigg\{h_{0,l}-h_{2,l}+\frac{1}{2}\psi_c^*\Gamma_{2,l} \\
&\qquad+\sum_{j=0}^{l-1}\Bigg[h_{0,j}\Gamma_{1,l-j-1}-h_{2,j}\Gamma^*_{1,l-j-1}+\frac{1}{2}h^*_{1,j}\Gamma_{2,l-j-1} \\
&\qquad\quad-\frac{1}{2}h_{3,j}\Gamma^*_{2,l-j-1}+\sum_{m=0}^{j}\left(\frac{h^*_{l-j+1,m}}{l-j+2}\Gamma_{l-j+2,j-m}\right)\Bigg] \\
&\qquad-\sum_{j=0}^{l-2}\sum_{m=0}^{j}\left[\frac{h_{l-j+2,m}}{l-j+1}\Gamma^*_{l-j+1,j-m}\right]\Bigg\}.
\end{aligned}
\tag{87}
$$

For $k=2$, from (70), the functions $I_{2,l}(v)$ are given by

$$
I_{2,0}(v)=\frac{i}{k_c}\frac{\partial}{\partial v}\psi_c \tag{88}
$$

for $l=0$, and for $l\geq 1$

$$
\begin{aligned}
I_{2,l}(v) = &-\sum_{j=0}^{l-1}\left[(2+j)p_{l-j}+jp^*_{l-j}\right]\,h_{2,j} \\
&+\frac{i}{k_c}\frac{\partial}{\partial v}\Bigg\{h_{1,l-1}+\psi_c\Gamma_{1,l-1}-h_{3,l-1}+\frac{1}{3}\psi_c^*\Gamma_{3,l-1}+\frac{1}{2}\sum_{j=0}^{l-1}h_{0,j}\Gamma_{2,l-j-1} \\
&\quad+\sum_{j=0}^{l-2}\Bigg[h_{1,j}\Gamma_{1,l-j-2}-h_{3,j}\Gamma^*_{1,l-j-2}-\frac{1}{2}h_{4,j}\Gamma^*_{2,l-j-2}+\frac{1}{3}h^*_{1,j}\Gamma_{3,l-j-2} \\
&\qquad\qquad+\sum_{m=0}^{j}\frac{h^*_{l-j,m}}{l-j+2}\Gamma_{l-j+2,j-m}\Bigg] \\
&\quad-\sum_{j=0}^{l-3}\frac{h_{5,j}}{3}\Gamma^*_{3,l-j-3}-\sum_{j=0}^{l-4}\sum_{m=0}^{j}\frac{h_{l-j+2,m}}{l-j}\Gamma^*_{l-j,j-m}\Bigg\}.
\end{aligned}
\tag{89}
$$

For $k>2$, the functions $I_{k,l}(v)$ are given by

$$
\begin{aligned}
I_{k,l}(v) = &-\sum_{j=0}^{l-1}\left[(k+j)p_{l-j}+j\,p^*_{l-j}\right]h_{k,j}(v) \\
&+\frac{i}{k_c}\frac{\partial}{\partial v}\Bigg\{h_{k-1,l}+\frac{\psi_c}{k-1}\Gamma_{k-1,l}+\sum_{l'=2}^{k-2}\sum_{j=0}^{l}\frac{h_{k-l',j}}{l'}\Gamma_{l',l-j} \\
&\qquad-h_{k+1,l-1}+\frac{\psi_c^*}{k+1}\Gamma_{k+1,l-1}. \\
&\qquad+\sum_{j=0}^{l-1}\left[h_{k-1,j}\Gamma_{1,l-j-1}+\frac{h_{1,j}}{k-1}\Gamma_{k-1,l-j-1}+\frac{h_{0,j}}{k}\Gamma_{k,l-j-1}\right]
\end{aligned}
\tag{90}
$$

$$+\sum_{j=0}^{l-2}\left[-h_{k+1,j}\Gamma^*_{1,l-j-2}+\frac{h^*_{1,j}}{k+1}\Gamma_{k+1,l-j-2}\right.$$
$$\left.\left.+\sum_{m=0}^{j}\left(\frac{h^*_{l-j,m}}{l+k-j}\Gamma_{l+k-j,j-m}-\frac{h_{k+l-j,m}}{l-j}\Gamma^*_{l-j,j-m}\right)\right]\right\};$$

in this last expression, if a subscript is negative the term is understood to be omitted, e.g. for $l=0$, $h_{k+1,l-1}$ is omitted.

The arguments $w_{k,l}$ to the resolvent (85) determine poles of $h_{k,l}(v)$ located at $v=z_{k,l}$ where $z_{k,l}\equiv iw_{k,l}/kk_c$. For $k\geq 1$, these poles always fall in the upper half plane above the phase velocity v_p:

$$z_{k,l}=z_0+\frac{i\gamma d_{k,l}}{k_c}=v_p+\frac{i\gamma(1+d_{k,l})}{k_c} \tag{91}$$

where $d_{k,l}\equiv 2(l+\delta_{k,1})/k$.

The relations (78) and (82) - (90) can be solved systematically to calculate p_j and $h_{k,l}$ to any order. The leading coefficient p_0 is determined by linear theory (77) and from the linear eigenfunction ψ_c one can also calculate $h_{0,0}$ and $h_{2,0}$, c.f. (83) and (88), respectively. These two coefficients then suffice to calculate p_1 from (78). From $\{p_1,h_{0,0},h_{2,0}\}$, the coefficients $h_{1,0}$ and $h_{3,0}$ can be determined and then $h_{0,1}$ and $h_{2,1}$. This provides the input to calculate p_2, and from this point on the structure of the calculation to all orders falls into a recognizable pattern that is summarized in Table I.

The expressions (82) and (85) for $h_{k,l}$ allow a useful evaluation of $\Gamma_{k,l}$ in terms of $I_{k,l}$. For $k=0$ the coefficient vanishes identically $\Gamma_{0,l}=0$, and for $k>0$, from (75) and (85), one has

$$\begin{aligned}
\Gamma_{k,l} &= \int_{-\infty}^{\infty} dv\, R_k(w_{k,l})\, I_{k,l}\\
&= \frac{1}{ikk_c}\left[1-\frac{1}{k^2\Lambda_k(z_{k,l})}\int_{-\infty}^{\infty} dv\,\frac{\eta(v,\mu)}{v-z_{k,l}}\right]\int_{-\infty}^{\infty} dv'\,\frac{I_{k,l}(v')}{v'-z_{k,l}}\\
&= \frac{1}{ikk_c}\left[\frac{k^2\Lambda_k(z_{k,l})-(\Lambda_1(z_{k,l})-1)}{k^2\Lambda_k(z_{k,l})}\right]\int_{-\infty}^{\infty} dv'\,\frac{I_{k,l}(v')}{v'-z_{k,l}}\\
&= -\left[\frac{ik/k_c}{k^2-1+\Lambda_1(z_{k,l})}\right]\int_{-\infty}^{\infty} dv'\,\frac{I_{k,l}(v')}{v'-z_{k,l}}
\end{aligned} \tag{92}$$

where in the last step the identity $k^2\Lambda_k(z_{k,l}) = k^2 - 1 + \Lambda_1(z_{k,l})$ has been used. The final expression in (92) will prove helpful in analyzing the form of $\Gamma_{k,l}$ as $\gamma \to 0^+$.

B. Analysis of the cubic coefficient

It is instructive at this point to evaluate and examine the cubic coefficient $p_1(\mu)$ in detail. From (78) we have

$$p_1 = -\frac{i}{k_c}\left[< \partial_v\tilde{\psi}_c, (h_{0,0} - h_{2,0}) > +\frac{1}{2} < \partial_v\tilde{\psi}_c, \psi_c^* > \Gamma_{2,0}\right], \tag{93}$$

so that $h_{0,0}$ and $h_{2,0}$ must be found from (82) and (85). A straightforward calculation yields

$$h_{0,0}(v) = \frac{1}{k_c^2}\frac{\partial}{\partial v}\left[\frac{-\eta(v,\mu)}{(v-z_0)\,(v-z_0^*)}\right] \tag{94}$$

$$h_{2,0}(v) = \frac{1}{2k_c^2}\left[\frac{\partial_v\psi_c}{(v-z_0)} + \frac{\Lambda_1^{(2)}(z_0)\,\eta(v,\mu)}{6(v-z_0)}\right] \tag{95}$$

where $\Lambda_1^{(2)}(z_0)$ is the second derivative defined in (32). Integrating these functions over velocity yields $\Gamma_{0,0} = 0$ and

$$\Gamma_{2,0} = \frac{-\Lambda_1^{(2)}(z_0)}{3\,k_c^2}. \tag{96}$$

The remaining integrals in (93) are expressible in terms of the derivatives of $\Lambda_1(z_0)$:

$$< \partial_v\tilde{\psi}_c, \psi_c^* > \ = \frac{ik_c}{2\gamma} \tag{97}$$

$$< \partial_v\tilde{\psi}_c, h_{0,0} > \ = -\frac{ik_c}{4\gamma^3}\left[1 - \frac{i\gamma\Lambda_1^{(2)}(z_0)}{k_c\Lambda_1'(z_0)} - \frac{2\gamma^2\Lambda_1^{(3)}(z_0)}{3k_c^2\Lambda_1'(z_0)}\right] \tag{98}$$

$$< \partial_v\tilde{\psi}_c, h_{2,0} > \ = \frac{2(\Lambda_1^{(2)}(z_0))^2 - 3\Lambda_1^{(4)}(z_0)}{48k_c^2\Lambda_1'(z_0)}. \tag{99}$$

Thus we find from (93)

$$\begin{aligned} p_1 = -\frac{1}{4\gamma^3}\Bigg[1 - \frac{i\gamma\Lambda_1^{(2)}(z_0)}{k_c\Lambda_1'(z_0)} - \frac{\gamma^2}{3k_c^2}\left(\frac{2\Lambda_1^{(3)}(z_0) - \Lambda_1'(z_0)\Lambda_1^{(2)}(z_0)}{\Lambda_1'(z_0)}\right) \\ + \frac{i\gamma^3}{12k_c^3\,\Lambda_1'(z_0)}\left(3\Lambda_1^{(4)}(z_0) - 2(\Lambda_1^{(2)}(z_0))^2\right)\Bigg] \end{aligned} \tag{100}$$

which simplifies to

$$p_1 = \frac{1}{\gamma^3}\left[-\frac{1}{4} + \mathcal{O}(\gamma)\right] \tag{101}$$

as $\gamma \to 0^+$.

The cubic coefficient is obviously singular in the weak growth rate regime and the strength of the singularity is set by the contribution from $h_{0,0}(v)$ which is the nonlinear correction to the equilibrium F_0 at this order. [39] The following summary of the calculation of $< \partial_v \tilde{\psi}_c, h_{0,0} >$ in (98) pinpoints the origin of the singularity. From (94) and the definition of $\tilde{\psi}_c$, one has

$$\begin{aligned}
< \partial_v \tilde{\psi}_c, h_{0,0} > &= \int_{-\infty}^{\infty} dv\, \frac{\partial \tilde{\psi}_c}{\partial v}^*(v) h_{0,0}(v) \\
&= \frac{1}{k_c^2 \Lambda_1'(z_0)} \int_{-\infty}^{\infty} dv\, \frac{1}{(v-z_0)^2} \frac{\partial}{\partial v}\left[\frac{-\eta(v,\mu)}{(v-z_0)\,(v-z_0^*)}\right] \\
&= -\frac{2}{k_c^2 \Lambda_1'(z_0)} \int_{-\infty}^{\infty} dv\, \frac{\eta(v,\mu)}{(v-z_0)^4\,(v-z_0^*)}.
\end{aligned} \tag{102}$$

Since $z_0 = v_p + i\gamma/k_c$ in (102), there is clearly a pinching singularity at v_p when $\gamma \to 0^+$. By expanding the integrand in partial fractions,

$$\begin{aligned}
\frac{1}{(v-z_0)^4\,(v-z_0^*)} = \frac{(k_c/2i\gamma)}{(v-z_0)^4} - \frac{(k_c/2i\gamma)^2}{(v-z_0)^3} + \frac{(k_c/2i\gamma)^3}{(v-z_0)^2} \\
- \left(\frac{k_c}{2i\gamma}\right)^4 \left[\frac{1}{v-z_0} - \frac{1}{v-z_0^*}\right],
\end{aligned} \tag{103}$$

(102) becomes

$$\begin{aligned}
< \partial_v \tilde{\psi}_c, h_{0,0} > = -\frac{2}{k_c^2 \Lambda_1'(z_0)} \Bigg\{ -\left(\frac{k_c}{2i\gamma}\right)^4 \left[(\Lambda_1(z_0)-1) - (\Lambda_1(z_0^*)-1)\right] \\
+ \left(\frac{k_c}{2i\gamma}\right)^3 \Lambda_1'(z_0) - \left(\frac{k_c}{2i\gamma}\right)^2 \frac{\Lambda_1^{(2)}(z_0)}{2} \\
+ \left(\frac{k_c}{2i\gamma}\right) \frac{\Lambda_1^{(3)}(z_0)}{6} \Bigg\}.
\end{aligned} \tag{104}$$

Since $\Lambda_1(z_0) = 0$ and $\Lambda_1(z_0^*) = 0$, the γ^{-4} terms vanish *identically* and do not contribute to the $\gamma \to 0^+$ limit; thus (104) reduces to (98) and a γ^{-3} singularity.

The interpretation of the γ^{-3} singularity is suggested by examining the form of the mode equation (72), truncated at cubic order, in the $\gamma \to 0^+$ limit:

$$\dot{A} = A\left[\lambda - \frac{1}{4\gamma^3}\left(1 - \frac{i\gamma\Lambda_1^{(2)}(z_0)}{k_c\Lambda_1'(z_0)} + \mathcal{O}(\gamma^2)\right)|A|^2 + \cdots\right] \tag{105}$$

where $\lambda = \gamma - i\omega$; in terms of amplitude and phase variables $A = \rho e^{-i\theta}$ this reads

$$\dot{\rho} = \rho\left[\gamma - \frac{1}{4\gamma^3}\rho^2 + \mathcal{O}(\rho^4)\right] \tag{106}$$

$$\dot{\theta} = \omega - \mathrm{Re}\left(\frac{\Lambda_1^{(2)}(z_0)}{\Lambda_1'(z_0)}\right)\frac{\rho^2}{4k_c\gamma^2} + \mathcal{O}(\rho^4). \tag{107}$$

The manifest singularities at $\gamma = 0$ can be removed by introducing a rescaled amplitude variable:

$$\rho(t) \equiv \gamma^2\, r(\gamma t) \tag{108}$$

which evolves on the time scale $\tau \equiv \gamma t$. In terms of $r(\tau)$, the mode equation becomes

$$\frac{dr}{d\tau} = r\left[1 - \frac{1}{4}r^2 + \mathcal{O}(\gamma^8 r^4)\right] \tag{109}$$

$$\frac{d\theta}{dt} = \omega - \frac{\gamma^2}{4k_c}\mathrm{Re}\left(\frac{\Lambda_1^{(2)}(z_0)}{\Lambda_1'(z_0)}\right)r^2 + \mathcal{O}(\gamma^8 r^4). \tag{110}$$

For the terms shown explicitly in (109) - (110), the $\gamma \to 0^+$ limit is now well behaved; this suggests that the effect of the singularities in the coefficients is to produce the scaling $\rho \sim \gamma^2$ in the dynamics of the mode amplitude. Our ansatz in (108) bears some resemblence to the simpler case of Hopf bifurcation discussed in the Introduction but the scaling is quantitatively different indicating a much stronger nonlinear effect and a smaller nonlinearly evolving mode.

It is also important to note that to this order the result (108) has a remarkable degree of universality. The γ^{-3} singularity in (101) sets the scaling in (108) and the coefficient, $-1/4$, of this singularity is completely independent of the underlying equilibrium $F_0(v, \mu_c)$. Thus, for example, the singularity is the same for a beam-plasma instablility (complex λ) as for a two-stream instability (real λ). Indeed to this order, the rescaled amplitude equation for $dr/d\tau$ is also completely independent of F_0 as $\gamma \to 0^+$.

These remarks tacitly assume that the neglected higher order terms in (105) do not alter conclusions reached for the truncated equations. The remainder of the paper is devoted to a systematic analysis of the higher order terms in the series for p and $h_{k,l}$. It will turn out that the coefficients p_j for $j \geq 2$ are also singular and that the naive estimate $\mathcal{O}(\gamma^8)$ in (109) and (110) is not correct. In fact the terms that are higher order in r are *not* higher order in γ, rather they appear to be as important for the dynamics of $A(t)$ as the cubic term $A|A|^2$. Nevertheless the singularities to *all* orders are absorbed by the scaling in (108). The dependence of the higher order terms on $F_0(v, \mu_c)$ as $\gamma \to 0^+$ will be discussed later.

IV. SINGULARITY STRUCTURE OF THE EXPANSION

The detailed calculation of p_j rapidly becomes prohibitively laborious, but the recursion relations determining the higher order coefficients in terms of lower order quantities (cf. Table I) can be analyzed to obtain useful information. Most basic is the question: for $j > 1$, how singular is p_j as $\gamma \to 0^+$? This issue requires an accurate estimate of the build up of pinching singularities in the integrals found in p_j. For this purpose I introduce an "index" which allows the divergence of a given integral to be assessed by a simple counting procedure.

A. Definition of the index

For $n > 0$ define

$$D_n(\alpha, v) \equiv \frac{1}{(v-\alpha_1)(v-\alpha_2)\cdots(v-\alpha_n)} \tag{111}$$

where $\alpha \equiv (\alpha_1, \ldots, \alpha_n)$ and define $D_0(\alpha, v) \equiv 1$. Evaluating p_j for $j \geq 1$ involves integrands of the form

$$\mathcal{G}(v, \mu) = D_m(\beta, v)^* \, D_n(\alpha, v) \, \frac{\partial^l \eta(v, \mu)}{\partial v^l} \tag{112}$$

with $m + n \geq 1$. The poles in (112) are given by

$$\alpha_j = z_0 + i\gamma\nu_j/k_c \qquad j = 1, \ldots, n \tag{113}$$

$$\beta_j^* = z_0^* - i\gamma\zeta_j/k_c \qquad j = 1, \ldots, m; \tag{114}$$

hence they lie along the vertical line Re $v = v_p$ at locations determined by the numbers $\nu_j \geq 0$ and $\zeta_j \geq 0$ which are assumed to be independent of F_0 for all j; in particular ν_j and ζ_j are independent of γ. The function $\mathcal{G}(v,\mu)$ depends on μ through z_0, γ, and $\eta(v,\mu)$.

The *index* of $\mathcal{G}(v,\mu)$ in (112) is defined to be

$$\text{Ind}\,[\mathcal{G}] \equiv m + n + l - 2. \tag{115}$$

Since we assume $m + n \geq 1$, Ind $[\mathcal{G}] \geq -1$. If $mn \neq 0$, then as $\gamma \to 0^+$, the integral of $\mathcal{G}$ diverges due to a pinching singularity at the phase velocity v_p. When Ind $[\mathcal{G}] \geq 0$, the index of $\mathcal{G}$ gives the maximum possible strength of this divergence.

Lemma IV.1 *For $\mathcal{G}(v,\mu)$ in (112) with $m + n \geq 2$ and $J =$ Ind $[\mathcal{G}]$, the integral of $\mathcal{G}$ satisfies*

$$\lim_{\gamma\to 0^+} \left|\gamma^J \int_{-\infty}^{\infty} dv\,\mathcal{G}(v,\mu)\right| < \infty. \tag{116}$$

If J is replaced by $J-1$, then the limit (116) diverges in general unless $mn = 0$ in which case the limit is zero for any $J > 0$.

Proof.

Since

$$\int_{-\infty}^{\infty} dv\, D_m(\beta,v)^*\, D_n(\alpha,v)\,\frac{\partial^l \eta(v,\mu)}{\partial v^l} = \tag{117}$$
$$\int_{-\infty}^{\infty} dv\,\frac{\partial^{l-1}\eta(v,\mu)}{\partial v^{l-1}}\left\{\sum_{j=1}^{m} D_{m+1}(\beta,\beta_j,v)^*\, D_n(\alpha,v) + \sum_{j=1}^{n} D_m(\beta,v)^*\, D_{n+1}(\alpha,\alpha_j,v)\right\},$$

we can reduce to the case with $l = 0$. If $mn = 0$ then there is no pinching singularity and the integral in (116) has a finite limit, cf. (32); hence the limit is zero if $J > 0$. Assume that $mn > 0$, then for integrals $\int_{-\infty}^{\infty} dv\, D_m(\beta,v)^*\, D_n(\alpha,v)\,\eta(v,\mu)$

with $m+n=2$ the integration can be easily done and (116) explicitly verified. For integrals with $m+n>2$, by expanding the integrand in partial fractions, they can be re-expressed in terms of integrals with $m+n-1$. A simple induction argument then establishes (116). In Appendix B the limit in (116) for $l=0$ is shown to be non-zero in general so, barring an accidental cancellation, if J is replaced by $J-1$ then the modified limit will diverge as γ^{-1}. $\square$

The definition in (115) can be generalized in some obvious ways without sacrificing its usefulness. Suppose $q(\mu)$ is a function of μ with the asymptotic behavior

$$\lim_{\gamma\to 0^+} q(\mu) \sim \gamma^{-\delta}, \tag{118}$$

then we define the index of $q(\mu)\,\mathcal{G}(v,\mu)$ to be

$$\text{Ind}\,[q\,\mathcal{G}] \equiv \text{Ind}\,[\mathcal{G}] + \delta. \tag{119}$$

Estimates of the form (116) still hold for $q\mathcal{G}$ with $J = \max\{\delta, \text{Ind}\,[q\,\mathcal{G}]\}$; the anomalous case $J=\delta$ arises when $\text{Ind}\,[\mathcal{G}] = -1$ since the integral of $\mathcal{G}$ is then nonsingular and the asymptotic behavior of $q\int dv\,\mathcal{G}$ is determined by $q(\mu)$. Finally if $\mathcal{G}_1(v,\mu)$ and $\mathcal{G}_2(v,\mu)$ have indices satisfying $\text{Ind}\,[\mathcal{G}_1] \geq \text{Ind}\,[\mathcal{G}_2]$ then we define the index of the sum to be the larger index:

$$\text{Ind}\,[\mathcal{G}_1 + \mathcal{G}_2] \equiv \text{Ind}\,[\mathcal{G}_1]. \tag{120}$$

B. Examples

As a simple example of the index, note that the functions ψ_c, $h_{0,0}$, and $h_{2,0}$ have indices given by

$$\text{Ind}\,[\psi_c] = -1 \tag{121}$$

$$\text{Ind}\,[h_{0,0}] = 1 \tag{122}$$

$$\text{Ind}\,[h_{2,0}] = 1; \tag{123}$$

similarly the integrands in (97) - (99) give

$$\text{Ind}\ [\partial_v \tilde{\psi}_c{}^* \psi_c^*] = 1 \tag{124}$$

$$\text{Ind}\ [\partial_v \tilde{\psi}_c{}^* h_{0,0}] = 3 \tag{125}$$

$$\text{Ind}\ [\partial_v \tilde{\psi}_c{}^* h_{2,0}] = 3. \tag{126}$$

From Lemma IV.1, this information immediately tells us that the singularity of p_1 cannot be worse than γ^{-3}.

A further useful observation is the effect of various operators on the index.

Lemma IV.2 *For $\mathcal{G}(v,\mu)$ as in* (112)*, the operators $\partial_v \mathcal{G}$, $\mathcal{P}_\perp \mathcal{G}$ and $R_k(w_{k,l})\mathcal{G}$ either leave the index unchanged or else increase it by one:*

$$\text{Ind}\ [\partial_v \mathcal{G}] = \text{Ind}\ [\mathcal{G}] + 1 \tag{127}$$

$$\text{Ind}\ [\mathcal{P}_\perp \mathcal{G}] = \text{Ind}\ [\mathcal{G}] \tag{128}$$

$$\text{Ind}\ [R_k(w_{k,l})\mathcal{G}] = \text{Ind}\ [\mathcal{G}] + 1. \tag{129}$$

Proof.

The first identity is obvious, and the second follows immediately from the definition of $\mathcal{P}_\perp$ below (64). For the third identity, the resolvent (81) gives

$$R_k(w_{k,l})\,\mathcal{G} = \frac{1}{ikk_c(v - z_{k,l})}\left[\mathcal{G}(v,\mu) - \frac{\eta(v,\mu)}{k^2\,\Lambda_k(z_{k,l})}\int_{-\infty}^{\infty} dv'\,\frac{\mathcal{G}(v',\mu)}{v' - z_{k,l}}\right]. \tag{130}$$

For the first term the index is increased by one, $\text{Ind}\ [\mathcal{G}/(v - z_{k,l})] = \text{Ind}\ [\mathcal{G}] + 1$. In the second term, $\text{Ind}\ [\eta(v,\mu)/(v - z_{k,l})] = -1$, so the function multiplying the integral

$$q(\mu) \equiv \frac{1}{k^2\Lambda_k(z_{k,l})}\int_{-\infty}^{\infty} dv'\,\frac{\mathcal{G}(v',\mu)}{v' - z_{k,l}} \tag{131}$$

will determine the index and the singular behavior of the corresponding velocity integral (recall the discussion below (119)). Since

$$\int_{-\infty}^{\infty} dv' \, \frac{\mathcal{G}(v',\mu)}{v'-z_{k,l}} \sim \frac{1}{\gamma^{(1+\text{Ind }[\mathcal{G}])}} \tag{132}$$

and

$$k^2\Lambda_k(z_{k,l}) = k^2 - 1 + \Lambda_1(z_{k,l}) = k^2 - 1 + \Lambda_1'(z_0)\left(\frac{i\gamma d_{k,l}}{k_c}\right) + \mathcal{O}(\gamma^2), \tag{133}$$

$q(\mu)$ has the asymptotic form

$$\lim_{\gamma\to 0^+} \left|\gamma^J \, q(\mu)\right| < \infty \tag{134}$$

with $J = \text{Ind }[\mathcal{G}] + 1 + \delta_{k,1}$. Hence the index of the second term in (130) is $\text{Ind }[\mathcal{G}] + 1$ and $\text{Ind }[\mathcal{G}]$ for $k = 1$ and $k \neq 1$, respectively. In either case, the overall index is given by (129).

□

C. Estimate of singularities

The main result on the singularity structure of the amplitude expansions for $p(\sigma,\mu)$ and $h_{k,l}(v,\sigma)$ can now be proved.

Theorem IV.1 *For $k \geq 0$ and $l \geq 0$, the indices of $I_{k,l}$ and $h_{k,l}$ are given by*

$$\text{Ind }[I_{k,l}] = \text{Ind }[h_{k,l}] - 1 \tag{135}$$

$$\text{Ind }[h_{k,l}] = 2k + 4l - 3 + 4(\delta_{k,0} + \delta_{k,1}). \tag{136}$$

The asymptotic behavior of the coefficients $\Gamma_{k,l}$ and p_j satisfies

$$\lim_{\gamma\to 0^+} \left|\gamma^J \, \Gamma_{k,l}\right| < \infty \qquad \text{where } \; J = \text{Ind }[h_{k,l}] + \delta_{k,1} \tag{137}$$

$$\lim_{\gamma\to 0^+} \left|\gamma^{4j-1} \, p_j(\mu)\right| < \infty \qquad \text{for } \; j \geq 1. \tag{138}$$

Proof.

1. Since $h_{0,l} = I_{0,l}/2\gamma(1+l)$ and $h_{k,l} = R_k(w_{k,l})I_{k,l}$, it follows immediately from (119) and (129) that the indices of $I_{k,l}$ and $h_{k,l}$ differ by one as stated in (135) - *provided* the index of $I_{k,l}$ is well defined. The induction argument outlined below shows that each function $I_{k,l}$ is obtained via the recursion relations as a sum of terms each having a well defined index. Thus $I_{k,l}$ will always have a well defined index.

2. The estimate in (137) follows from the formula for $\Gamma_{k,l}$ in (92):

$$\Gamma_{k,l} = -\frac{(ik/k_c)}{k^2 - 1 + \Lambda_1(z_{k,l})} \int_{-\infty}^{\infty} dv\, \frac{I_{k,l}(v)}{v - z_{k,l}}; \tag{139}$$

as $\gamma \to 0^+$, the integral cannot diverge more strongly than

$$\int_{-\infty}^{\infty} dv\, \frac{I_{k,l}(v)}{v - z_{k,l}} \sim \left(\frac{1}{\gamma}\right)^{\text{Ind }[h_{k,l}]}, \tag{140}$$

and the function multiplying the integral is nonsingular unless $k = 1$ in which case it is $\mathcal{O}(\gamma^{-1})$.

3. The index formula for $h_{k,l}$ (136) and the estimate for $p_j(\mu)$ in (138) are proved by induction using the recursion relations. The induction argument is organized by the pattern shown in Table I. At the top of Table I, the results in (136) and (138) have been explicitly verified for $\{h_{0,0}, h_{2,0}\}$ and p_1 in Section III.B.

4. Moving downward in Table I, the next quantities to consider are $h_{1,0}$ and $h_{3,0}$ from (87) and (90), respectively:

$$h_{1,0} = \frac{i}{k_c} R_1(w_{1,0})\, \mathcal{P}_\perp \frac{\partial}{\partial v} \left\{ h_{0,0} - h_{2,0} + \frac{1}{2}\psi_c^* \Gamma_{2,0} \right\} \tag{141}$$

$$h_{3,0} = \frac{i}{k_c} R_3(w_{3,0})\, \frac{\partial}{\partial v} \left\{ h_{2,0} + \frac{\psi_c}{2} \Gamma_{2,0} \right\}. \tag{142}$$

Reading right to left in these expressions, the index for each of the quantities in braces $\{\cdots\}$ is 1, and this index is increased to 2 by ∂_v and further increased to 3 by the resolvent; hence Ind $[h_{1,0}] = 3$ and Ind $[h_{3,0}] = 3$ in

agreement with (136). Next, according to Table I, are $h_{0,1}$ and $h_{2,1}$; from (82) and (89) they are:

$$h_{0,1} = \frac{1}{4\gamma}\Bigg[-(p_1+p_1^*)\,h_{0,0}(v) \\ +\frac{i}{k_c}\frac{\partial}{\partial v}\left\{\left[h_{1,0}^* - \psi_c\Gamma_{1,0}^* + \frac{h_{0,0}^*}{2}\Gamma_{2,0}\right] - cc\right\}\Bigg] \tag{143}$$

$$h_{2,1} = R_2(w_{2,1})\Bigg[-2p_1\,h_{2,0} \\ +\frac{i}{k_c}\frac{\partial}{\partial v}\left\{h_{1,0} + \psi_c\Gamma_{1,0} - h_{3,0} + \frac{1}{3}\psi_c^*\Gamma_{3,0} + \frac{1}{2}h_{0,0}\Gamma_{2,0}\right\}\Bigg]. \tag{144}$$

For $h_{0,1}$ the expression in braces $\{\cdots\}$ has index 3 and the larger expression in brackets $[\cdots]$ has index 4, so Ind $[h_{0,1}] = 5$ in agreement with (136). Similarly the index of $h_{2,1}$ is seen to have the correct value Ind $[h_{2,1}] = 5$.

5. This provides enough information to verify (138) for p_2. From (78) this coefficient is $p_2 = i[\mathcal{A}_2 + \mathcal{B}_2]/k_c$ where

$$\mathcal{A}_2 = -\Bigg[< \partial_v\tilde{\psi}_c, (h_{0,1} - h_{2,1}) > + \frac{1}{2} < \partial_v\tilde{\psi}_c, \psi_c^* > \Gamma_{2,1} \\ + < \partial_v\tilde{\psi}_c, h_{0,0} > \Gamma_{1,0} - < \partial_v\tilde{\psi}_c, h_{2,0} >\Bigg] \tag{145}$$

$$\mathcal{B}_2 = -\Bigg[\frac{1}{2} < \partial_v\tilde{\psi}_c, h_{1,0}^* > \Gamma_{2,0} + \frac{\Gamma_{3,0}}{3} < \partial_v\tilde{\psi}_c, h_{2,0}^* > \\ -\frac{\Gamma_{2,0}^*}{2} < \partial_v\tilde{\psi}_c, h_{3,0} >\Bigg]. \tag{146}$$

Since multiplying by $\partial_v\tilde{\psi}_c{}^*$ increases the index of an integrand by two, cf. (121) - (126), it is straightforward to verify that p_2 satisfies the estimate in (138). For example, consider the first term in (145). Since Ind $[h_{0,1}] =$ Ind $[h_{2,1}] = 5$, the integrand has index Ind $[\partial_v\tilde{\psi}_c{}^*(h_{0,1} - h_{2,1})] = 7$ so the worst possible singularity for the integral is

$$< \partial_v\tilde{\psi}_c, (h_{0,1} - h_{2,1}) > \sim \frac{1}{\gamma^7} \tag{147}$$

which is consistent with (138). A similar verification for the remaining terms in p_2 is straightforward.

6. According to Table I, for fixed $N \geq 1$, given the functions

$$\{h_{k,l}(v) \mid k = 0, \ldots, N+1 \ \text{ and } \ l = 0, \ldots, l_{max}(N,k)\} \tag{148}$$

where $l_{max}(N,k) \equiv N - k + 1 - 2(\delta_{k,0} + \delta_{k,1})$ and the lower order coefficients

$$\{p_j \mid j = 1, \ldots, N\}, \tag{149}$$

then one can calculate the additional functions

$$\{h_{k,l}(v) \mid k = 0, \ldots, N+2 \ \text{ for } \ l = l_{max}(N+1,k)\} \tag{150}$$

and the next coefficient p_{N+1}. For $N = 2$, it has been shown that the functions (148) and the coefficients (149) satisfy (136) and (138), respectively.

7. The assumption for the induction step is now clear. For arbitrary $N \geq 2$, assume that the functions (148) satisfy (136) and also that the coefficients (149) satisfy the estimate in (138). Then from this hypothesis, the recursion relations determining the functions in (150) and for p_{N+1} imply that these new quantities will also satisfy (136) and (138), respectively. Verifying this conclusion involves only a routine evaluation of the indices arising from the recursion relations as was done above for $N = 2$. This completes the proof of (136) and (138).

□

D. Implications of Theorem IV.1

The index formula for $h_{k,l}(v)$ and the asymptotic estimate on $\Gamma_{k,l}$ imply that in the the general expression for p_j (78) the dominant terms are in $\mathcal{A}_j$. More precisely, one finds $\mathcal{A}_j \sim \gamma^{-(4j-1)}$ and $\mathcal{B}_j \sim \gamma^{-(4j-2)}$ as $\gamma \to 0^+$. The estimates in (137) and (138) motivate the following asymptotic definitions for $\Gamma_{k,l}$ and p_j:

$$c_{k,l}(\mu_c) \equiv \lim_{\gamma\to 0^+} \left[\gamma^{(2k+4l-3+5\delta_{k,1})}\,\Gamma_{k,l}\right] \qquad \text{for } \; k \geq 1 \; \text{ and } \; l \geq 0 \tag{151}$$

$$b_j(\mu_c) \equiv \lim_{\gamma\to 0^+} \left[\gamma^{4j-1}\, p_j(\mu)\right] = \lim_{\gamma\to 0^+} \left[\gamma^{4j-1}\, \mathcal{A}_j(\mu)\right] \qquad \text{for } \; j \geq 1. \tag{152}$$

In this notation, the asymptotic form of Γ_k for $k \geq 1$ is

$$\Gamma_k = \left(\frac{1}{\gamma}\right)^{2k-3+5\delta_{k,1}} \left[\Gamma_k^c + \mathcal{O}(\gamma)\right] \tag{153}$$

where

$$\Gamma_k^c(\mu_c) \equiv \sum_{l=0}^{\infty} c_{k,l}(\mu_c) r^{2l}. \tag{154}$$

1. Mode amplitude equation

The full mode amplitude equation,

$$\frac{dA}{dt} = \lambda A + A \sum_{j=1}^{\infty} p_j(\mu)\sigma^j \tag{155}$$

when written in polar variables $A = \rho\, e^{-i\theta}$ gives

$$\frac{d\rho}{dt} = \rho\gamma + \rho \sum_{j=1}^{\infty} \left[\mathrm{Re}\; p_j(\mu)\right] \rho^{2j} \tag{156}$$

$$\frac{d\theta}{dt} = \omega - \sum_{j=1}^{\infty} \left[\mathrm{Im}\; p_j(\mu)\right]\, \rho^{2j}. \tag{157}$$

The $\gamma \to 0^+$ limit can be taken using the asymptotic behavior for p_j (152) given by Theorem IV.1 and the rescaling $\rho(t) = \gamma^2\, r(\gamma t)$ suggested by the low order analysis (108):

$$\frac{dr}{d\tau} = r \left\{ 1 + \sum_{j=1}^{\infty} \left[\mathrm{Re}\; b_j(\mu_c) + \mathcal{O}(\gamma)\right]\, r^{2j} \right\} \tag{158}$$

$$\frac{d\theta}{dt} = \omega - \gamma \sum_{j=1}^{\infty} \left[\mathrm{Im}\; b_j(\mu_c) + \mathcal{O}(\gamma)\right]\, r^{2j} \tag{159}$$

where $\omega = k_c v_p$ is the linear mode frequency. These equations are non-singular at every order as $\gamma \to 0^+$, and the final asymptotic equations are

$$\frac{dr}{d\tau} = rR(r, \mu_c) \tag{160}$$

$$\frac{d\theta}{dt} = \omega, \tag{161}$$

where

$$R(r, \mu_c) \equiv \left\{ 1 + \sum_{j=1}^{\infty} \left[\text{Re } b_j(\mu_c) \right] r^{2j} \right\}. \tag{162}$$

If the equilibrium is reflection-symmetric, then $\omega = 0$ and $[\text{Im } b_j(\mu_c) + \mathcal{O}(\gamma)] = 0$ so the phase equation (159) becomes $d\theta/dt = 0$.

Note that, in contrast to the rescaled amplitude equation for Hopf bifurcation (3), in this case the higher order terms in r are not higher order in γ; thus there is no apparent justification for a truncation of the expansion after the leading nonlinear terms. The dependence of $R(r, \mu_c)$ on μ_c is analyzed in Section V.

2. Electric field

The electric field,

$$E(x, t) = \sum_{k=1}^{\infty} (e^{ikk_c x} \, E_k(t) + \, cc), \tag{163}$$

is obtained from the potential $E = -\partial_x \Phi$, and from Poisson's equation (13) the Fourier components are

$$ikk_c E_k(t) = -\int_{-\infty}^{\infty} dv \; f_k^u(v, t). \tag{164}$$

With (52) for $f_k^u(v, t)$ this becomes

$$E_k(t) = \begin{cases} \frac{i}{k_c} A(1 + \sigma\Gamma_1) & k = 1 \\ \\ \frac{i}{kk_c} A^k \, \Gamma_k & k > 1; \end{cases} \tag{165}$$

hence the field is given by

$$E(x, t) = \left(\frac{i}{k_c} \left[\gamma^2 \, r(\tau) \, (1 + \gamma^4 \, r^4 \Gamma_1) e^{i(k_c x - \theta(t))} + \sum_{k=2}^{\infty} \frac{\gamma^{2k} \Gamma_k \, r^k}{k} e^{ik(k_c x - \theta(t))} \right] + cc \right). \tag{166}$$

In the $\gamma \to 0^+$ limit, this gives

$$E(x,t) = \frac{i\gamma^2}{k_c}\left[r(\tau)\,\left(1+\Gamma_1^c r(\tau)^4 + \mathcal{O}(\gamma)\right) e^{i(k_c x - \theta(t))} \right. \tag{167}$$

$$\left. + \gamma \sum_{k=2}^{\infty} \frac{\left(\Gamma_k^c\, r(\tau)^k + \mathcal{O}(\gamma)\right)}{k} e^{ik(k_c x - \theta(t))}\right] + cc$$

where Γ_k^c is defined in (154). This result describes an electric field that obeys trapping scaling $E \sim \gamma^2$ and also indicates that all higher spatial harmonics are uniformly $\mathcal{O}(\gamma)$ relative to the wavenumber of the unstable mode.

V. ASYMPTOTIC RECURSION RELATIONS

From the definition of b_j (152) one expects these limits to depend on the parameters μ_c through the underlying equilibrium $F_0(v,\mu_c)$. A striking feature of the explicitly calculated cubic coefficient (101) was the result

$$b_1 = -\frac{1}{4}; \tag{168}$$

thus b_1 is a pure number and independent of μ_c. It is natural to ask what happens for the higher order singularities. The dependence of b_j on μ_c can be systematically investigated by taking the $\gamma \to 0^+$ limit of the recursion relations directly rather than calculating individual coefficients p_j and studying their asymptotic behavior.

Let $\hat{h}_{k,l}$ and $\hat{I}_{k,l}$ denote the terms of maximum index in $h_{k,l}$ and $I_{k,l}$ respectively. From Theorem IV.1, this definition implies

$$h_{k,l} = \hat{h}_{k,l} + \text{ [terms of index } 2k+4l-4+4(\delta_{k,0}+\delta_{k,1}) \text{ or less]} \tag{169}$$

$$I_{k,l} = \hat{I}_{k,l} + \text{ [terms of index } 2k+4l-5+4(\delta_{k,0}+\delta_{k,1}) \text{ or less]} \tag{170}$$

Since only terms of maximum index can contribute to the limit in (152), the expression for p_j in (78) yields an explicit formula for b_j

$$ik_c\, b_j = \lim_{\gamma\to 0^+} \gamma^{4j-1}\left[< \partial_v\tilde{\psi}_c, (\hat{h}_{0,j-1} - \hat{h}_{2,j-1}) > \right. \tag{171}$$

$$\left. + \sum_{l=0}^{j-2}\left(< \partial_v\tilde{\psi}_c, \hat{h}_{0,j-l-2} > \Gamma_{1,l} - < \partial_v\tilde{\psi}_c, \hat{h}_{2,j-l-2} > \Gamma_{1,l}^*\right)\right].$$

Thus to study b_j, only recursion relations for $\hat{h}_{k,l}$ are required.

A. Truncated recursion relations

From (82) - (90), these relations are obtained by simply discarding terms of submaximal index. This truncation yields the following expressions:

$$\hat{h}_{0,l} = \frac{\hat{I}_{0,l}}{(1+l)(\lambda+\lambda^*)} \tag{172}$$

for $k = 0$, with

$$\hat{I}_{0,l}(v) = \begin{cases} \frac{i}{k_c}\frac{\partial}{\partial v}(\psi_c^* - \psi_c) & l = 0 \\ \\ -\sum_{j=0}^{l-1}(1+j)(p_{l-j}+p_{l-j}^*)\,\hat{h}_{0,j}(v) & l > 0 \\ \quad +\frac{i}{k_c}\frac{\partial}{\partial v}\left\{\left(\hat{h}_{1,l-1}^* - \psi_c\Gamma_{1,l-1}^* + \sum_{j=0}^{l-2}\hat{h}_{1,j}^*\Gamma_{1,l-j-2}\right) - cc\right\} \end{cases} \tag{173}$$

For $k \geq 1$, we obtain $\hat{h}_{k,l}(v)$ by calculating $R_k(w_{k,l})\,\hat{I}_{k,l}$ and discarding any terms of index less than Ind $[h_{k,l}]$. The required forms for $\hat{I}_{k,l}$ are

$$\hat{I}_{1,l}(v) = \frac{i}{k_c}\mathcal{P}_\perp\frac{\partial}{\partial v}\left\{\hat{h}_{0,l} - \hat{h}_{2,l} + \sum_{j=0}^{l-1}\left(\hat{h}_{0,j}\Gamma_{1,l-j-1} - \hat{h}_{2,j}\Gamma_{1,l-j-1}^*\right)\right\} \\ - \sum_{j=0}^{l-1}\left[(2+j)p_{l-j} + (1+j)p_{l-j}^*\right]\hat{h}_{1,j}. \tag{174}$$

$$\hat{I}_{2,l}(v) = \begin{cases} \frac{i}{k_c}\frac{\partial}{\partial v}\psi_c & l = 0 \\ \\ -\sum_{j=0}^{l-1}\left[(2+j)p_{l-j} + jp_{l-j}^*\right]\,\hat{h}_{2,j} & l > 0 \\ \qquad +\frac{i}{k_c}\frac{\partial}{\partial v}\left\{\hat{h}_{1,l-1} - \hat{h}_{3,l-1} + \psi_c\Gamma_{1,l-1}\right. \\ \qquad\qquad \left. + \sum_{j=0}^{l-2}\left[\hat{h}_{1,j}\Gamma_{1,l-j-2} - \hat{h}_{3,j}\Gamma_{1,l-j-2}^*\right]\right\} \end{cases} \tag{175}$$

and for $k \geq 3$

$$\hat{I}_{k,l}(v) = \frac{i}{k_c}\frac{\partial}{\partial v}\left\{\hat{h}_{k-1,l} - \hat{h}_{k+1,l-1} + \sum_{j=0}^{l-1}\hat{h}_{k-1,j}\Gamma_{1,l-j-1} - \sum_{j=0}^{l-2}\hat{h}_{k+1,j}\Gamma_{1,l-j-2}^*\right\} \\ - \sum_{j=0}^{l-1}\left[(k+j)p_{l-j} + j\,p_{l-j}^*\right]\hat{h}_{k,j}(v). \tag{176}$$

B. Integrated recursion relations

The expression for b_j (171) depends only on certain integrals of $\hat{h}_{k,l}$ which can be obtained directly from an appropriate integrated form of the truncated relations. For non-negative integers (m,n), consider limits of the form:

$$\mathcal{S}(m,n;\beta,\alpha) \equiv \left(\frac{i}{k_c}\right)^{m+n-2} \lim_{\gamma\to 0^+} \left(\frac{\gamma^{m+n-2}}{\Lambda_1'(z_0)} \int_{-\infty}^{\infty} dv\, D_m(\beta,v)^* \, D_n(\alpha,v)\, \eta(v,\mu)\right) \tag{177}$$

$$\mathcal{B}_{k,l}(m,n;\beta,\alpha) \equiv \left(\frac{i}{k_c}\right)^{m+n-1} \lim_{\gamma\to 0^+} \left(\frac{\gamma^{J}}{\Lambda_1'(z_0)} \int_{-\infty}^{\infty} dv\, D_m(\beta,v)^* \, D_n(\alpha,v)\, \hat{h}_{k,l}(v)\right) \tag{178}$$

$$\mathcal{C}_{k,l}(m,n;\beta,\alpha) \equiv \left(\frac{i}{k_c}\right)^{m+n-1} \lim_{\gamma\to 0^+} \left(\frac{\gamma^{J}}{\Lambda_1'(z_0)} \int_{-\infty}^{\infty} dv\, D_m(\beta,v)^* \, D_n(\alpha,v)\, \hat{h}_{k,l}(v)^*\right) \tag{179}$$

where $J \equiv \mathrm{Ind}\,[h_{k,l}]+m+n$ and $m+n \geq 2$ is required in (177). The resonance denominators in $D_m(\beta,v)^*\, D_n(\alpha,v)$ are defined using our previous notation (111), (113), and (114); in particular, the pole locations are:

$$\alpha_j = z_0 + i\gamma\nu_j/k_c \qquad\qquad j = 1,\ldots,n \tag{180}$$

$$\beta_j^* = z_0^* - i\gamma\zeta_j/k_c \qquad\qquad j = 1,\ldots,m, \tag{181}$$

and the sets of poles are indicated with the notation $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\beta = (\beta_1,\ldots,\beta_m)$, $\nu = (\nu_1,\ldots,\nu_n)$, and $\zeta = (\zeta_1,\ldots,\zeta_m)$. In using the notation above, e.g. for $\mathcal{S}(m,n;\beta,\alpha)$, it is understood that the first m arguments after the semi-colon correspond to the poles in the lower half-plane $D_m(\beta,v)^*$ and the remaining n arguments denote the poles in the upper half-plane $D_n(\alpha,v)$.

The latter two limits are related by

$$\mathcal{C}_{k,l}(m,n;\beta,\alpha) = e^{i\xi}\, [\mathcal{B}_{k,l}(n,m;\alpha,\beta)]^*(-1)^{m+n-1} \tag{182}$$

where the phase $e^{i\xi}$ is defined by

$$e^{i\xi(\mu_c)} \equiv \lim_{\gamma\to 0^+} \left(\frac{\Lambda_1'(z_0)^*}{\Lambda_1'(z_0)}\right). \tag{183}$$

Note that since $\Lambda_k'(z_0) = \epsilon_k'(z_0)$ for $\gamma \geq 0$, this is the phase mentioned in the Introduction.

From (171), the coefficients of the leading singularities are given by

$$b_j = \mathcal{B}_{2,j-1}(0,2;z_0,z_0) - \mathcal{B}_{0,j-1}(0,2;z_0,z_0) \\ + \sum_{l=0}^{j-2} [c_{1,l}^* \, \mathcal{B}_{2,j-l-2}(0,2;z_0,z_0) - c_{1,l} \, \mathcal{B}_{0,j-l-2}(0,2;z_0,z_0)] \tag{184}$$

where $c_{1,l}$ is defined from (151) by the limit

$$c_{1,l} = \lim_{\gamma \to 0^+} \left(\gamma^{4(l+1)} \, \Gamma_{1,l} \right). \tag{185}$$

This expression is evaluated below, c.f. (210).

The properties of $\mathcal{S}(m,n;\beta,\alpha)$ in (177) are analyzed in Appendix B; the main conclusion needed here concerns the dependence on μ_c.

Lemma V.1 *For non-negative integers (m,n) such that $m+n \geq 2$, $\mathcal{S}(m,n;\beta,\alpha)$ has the form*

$$\mathcal{S}(m,n;\beta,\alpha) = d(m,n;\zeta,\nu) + (-1)^{m+n} d(n,m;\nu,\zeta) \, e^{i\xi(\mu_c)} \tag{186}$$

where the real-valued functions $d(m,n;\zeta,\nu)$ are independent of $F_0(v,\mu_c)$. Thus $\mathcal{S}(m,n;\alpha,\beta)$ depends on μ_c only through the phase $\exp(i\xi)$*.*

Proof. See Appendix B. □

The analysis of $\mathcal{B}_{k,l}(m,n;\beta,\alpha)$ follows the pattern for the calculation of $h_{k,l}(v)$ in Table I. From the definition (178), $\mathcal{B}_{0,0}(m,n;\beta,\alpha)$ and $\mathcal{B}_{2,0}(m,n;\beta,\alpha)$ can be evaluated in terms of $\mathcal{S}(m,n;\alpha,\beta)$:

$$\mathcal{B}_{0,0}(m,n;\beta,\alpha) = \sum_{i=1}^{m} \mathcal{S}(m+2,n+1;\beta,\beta_i,z_0,\alpha,z_0) \\ + \sum_{i=1}^{n} \mathcal{S}(m+1,n+2;\beta,z_0,\alpha,\alpha_i,z_0) \tag{187}$$

$$\mathcal{B}_{2,0}(m,n;\beta,\alpha) = \frac{1}{2} \left[\sum_{i=1}^{m} \mathcal{S}(m+1,n+2;\beta,\beta_i,\alpha',z_0) \right. \\ \left. + \sum_{i=1}^{n+1} \mathcal{S}(m,n+3;\beta,\alpha',\alpha_i',z_0) \right] \tag{188}$$

where $\alpha' = (\alpha, z_{2,0})$. This expression for $\mathcal{B}_{2,0}(m,n;\beta,\alpha)$ uses the identities:

$$\lim_{\gamma \to 0^+} \Lambda_k(z_{k,l}) = \frac{k^2-1}{k^2} \tag{189}$$

from (33), and

$$\lim_{\gamma\to 0^+}\left(\gamma^{m+n}\int_{-\infty}^{\infty} dv\,\frac{D_m^*\,D_n}{v-z_{k,l}}\eta(v,\mu)\right) = -\delta_{m,0}\,\delta_{n,0}. \tag{190}$$

The latter identity is verified by noting that the integrand has index $m+n-1$ so if $m+n>0$ we get zero, and when $m=n=0$ then the integral reduces to $\Lambda_1(z_{k,l})-1$. From these expressions for $\mathcal{B}_{0,0}$ and $\mathcal{B}_{2,0}$ their dependence on μ_c is easily characterized.

Lemma V.2 *For all non-negative integers* (m,n)*,* $\mathcal{B}_{0,0}(m,n;\beta,\alpha)$ *and* $\mathcal{B}_{2,0}(m,n;\beta,\alpha)$ *depend on* μ_c *only through the phase* $\exp(i\xi)$*.*

Proof.

This follows immediately from (187) - (188) and Lemma V.1. □

Recursion relations for the remaining limits $\mathcal{B}_{k,l}(m,n;\beta,\alpha)$ and also for $c_{k,l}$ in (151) follow from the truncated relations in (172) - (176); this derivation is briefly summarized for $k=0,1,2$, and $k\geq 3$. These relations are somewhat complicated, but provide the basis for Theorem V.1 below.

1. $\mathcal{B}_{0,l}(m,n;\beta,\alpha)$ *for* $l\geq 1$

From the definition (178) and substituting for $\hat{h}_{0,l}(v)$ from (172)

$$\mathcal{B}_{0,l}(m,n;\beta,\alpha) = \left(\frac{i}{k_c}\right)^{m+n-1}\lim_{\gamma\to 0^+}\left(\frac{\gamma^{4l+m+n+1}}{2\gamma(l+1)\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\,D_m(\beta,v)^*\,D_n(\alpha,v)\,\hat{I}_{0,l}(v)\right) \tag{191}$$

Using (173) the integral becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty} dv\,D_m(\beta,v)^*\,D_n(\alpha,v)\,\hat{I}_{0,l}(v) = \\
&-\sum_{j=0}^{l-1}(1+j)(p_{l-j}+p_{l-j}^*)\int_{-\infty}^{\infty} dv\,D_m(\beta,v)^*\,D_n(\alpha,v)\,\hat{h}_{0,j}(v) \\
&-\frac{i}{k_c}\int_{-\infty}^{\infty} dv\,\frac{\partial}{\partial v}(D_m^*\,D_n)\left[\left(\hat{h}_{1,l-1}^*-\psi_c\Gamma_{1,l-1}^*+\sum_{j=0}^{l-2}\hat{h}_{1,j}^*\Gamma_{1,l-j-2}\right)-cc\right]
\end{aligned} \tag{192}$$

With the substitution

$$-\frac{\partial}{\partial v}(D_m^*\, D_n) = \sum_{i=1}^{m} \frac{D_m^*\, D_n}{v-\beta_i^*} + \sum_{i=1}^{n} \frac{D_m^*\, D_n}{v-\alpha_i}, \tag{193}$$

the limit in (191) can be rewritten as

$$\begin{aligned}
&2(l+1)\mathcal{B}_{0,l}(m,n;\beta,\alpha) = \qquad\qquad (194)\\
&-\sum_{j=0}^{l-1}(1+j)\left[\lim_{\gamma\to 0^+}\gamma^{4(l-j)-1}(p_{l-j}+p_{l-j}^*)\right]\left(\frac{i}{k_c}\right)^{m+n-1}\lim_{\gamma\to 0^+}\left(\frac{\gamma^{4j+m+n+1}}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\, D_m^*\, D_n\, \hat{h}_{0,j}(v)\right)\\
&\qquad + \left(\frac{i}{k_c}\right)^{m+n}\sum_{i=1}^{m}\lim_{\gamma\to 0^+}\left(\frac{\gamma^{4l+m+n}}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\, \frac{D_m^*\, D_n}{v-\beta_i^*}\left[\hat{h}_{1,l-1}^* - \psi_c\Gamma_{1,l-1}^* + \cdots\right]\right)\\
&\qquad + \left(\frac{i}{k_c}\right)^{m+n}\sum_{i=1}^{n}\lim_{\gamma\to 0^+}\left(\frac{\gamma^{4l+m+n}}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\, \frac{D_m^*\, D_n}{v-\alpha_i}\left[\hat{h}_{1,l-1}^* - \psi_c\Gamma_{1,l-1}^* + \cdots\right]\right),
\end{aligned}$$

and then evaluated to obtain

$$\begin{aligned}
2(l+1)\mathcal{B}_{0,l}(m,n;\beta,\alpha) = &-\sum_{j=0}^{l-1}(1+j)(b_{l-j}+b_{l-j}^*)\,\mathcal{B}_{0,j}(m,n;\beta,\alpha)\\
&+\sum_{i=1}^{m}\Bigg\{\mathcal{C}_{1,l-1}(m+1,n;\beta,\beta_i,\alpha) - \mathcal{B}_{1,l-1}(m+1,n;\beta,\beta_i,\alpha)\\
&\quad -c_{1,l-1}^*\,\mathcal{S}(m+1,n+1;\beta,\beta_i,z_0,\alpha) + c_{1,l-1}\,\mathcal{S}(m+2,n;\beta,\beta_i,z_0,\alpha)\\
&\quad +\sum_{j=0}^{l-2}\left[c_{1,l-j-2}\,\mathcal{C}_{1,j}(m+1,n;\beta,\beta_i,\alpha) - c_{1,l-j-2}^*\,\mathcal{B}_{1,j}(m+1,n;\beta,\beta_i,\alpha)\right]\Bigg\}\\
&+\sum_{i=1}^{n}\Bigg\{\mathcal{C}_{1,l-1}(m,n+1;\beta,\alpha,\alpha_i) - \mathcal{B}_{1,l-1}(m,n+1;\beta,\alpha,\alpha_i)\\
&\quad -c_{1,l-1}^*\,\mathcal{S}(m,n+2;\beta,z_0,\alpha,\alpha_i) + c_{1,l-1}\,\mathcal{S}(m+1,n+1;\beta,z_0,\alpha,\alpha_i)\\
&\quad +\sum_{j=0}^{l-2}\left[c_{1,l-j-2}\,\mathcal{C}_{1,j}(m,n+1;\beta,\alpha,\alpha_i) - c_{1,l-j-2}^*\,\mathcal{B}_{1,j}(m,n+1;\beta,\alpha,\alpha_i)\right]\Bigg\}.
\end{aligned} \tag{195}$$

2. $\mathcal{B}_{1,l}(m,n;\beta,\alpha)$ for $l \geq 0$

From the definition (178)

$$\mathcal{B}_{1,l}(m,n;\beta,\alpha) = \left(\frac{i}{k_c}\right)^{m+n-1}\lim_{\gamma\to 0^+}\left(\frac{\gamma^{4l+m+n+3}}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\, D_m(\beta,v)^*\, D_n(\alpha,v)\,\hat{h}_{1,l}(v)\right). \tag{196}$$

Using $\hat{h}_{1,l} = R_1(w_{1,l})\,\hat{I}_{1,l}$ and (92) for $\Gamma_{1,l}$, this can be rewritten as

$$\mathcal{B}_{1,l}(m,n;\beta,\alpha) = -\,c_{1,l}\,\mathcal{S}(m,n+1;\beta,\alpha,z_{1,l}) \tag{197}$$
$$- \left(\frac{i}{k_c}\right)^{m+n} \lim_{\gamma\to 0^+} \left(\frac{\gamma^{4l+m+n+3}}{\Lambda_1'(z_0)} \int_{-\infty}^{\infty} dv\, \frac{D_m(\beta,v)^*\, D_n(\alpha,v)\, \hat{I}_{1,l}(v)}{v - z_{1,l}} \right)$$

where we have used (185). Using the expression for $\hat{I}_{1,l}(v)$ in (174) we obtain for the second term

$$\left(\frac{i}{k_c}\right)^{m+n} \lim_{\gamma\to 0^+} \left(\frac{\gamma^{4l+m+n+3}}{\Lambda_1'(z_0)} \int_{-\infty}^{\infty} dv\, \frac{D_m(\beta,v)^*\, D_n(\alpha,v)\, \hat{I}_{1,l}(v)}{v - z_{1,l}} \right) = \tag{198}$$
$$-\sum_{j=0}^{l-1} \Big[(2+j)b_{l-j} + (1+j)b_{l-j}^*\Big]\, \mathcal{B}_{1,j}(m,n+1;\beta,\alpha,z_{1,l})$$
$$+ \Big[\mathcal{B}_{0,l}(m,n+2;\beta,\alpha,z_{1,l},z_{1,l}) - \mathcal{B}_{2,l}(m,n+2;\beta,\alpha,z_{1,l},z_{1,l})$$
$$+ \sum_{j=0}^{l-1} \Big(c_{1,l-j-1}\mathcal{B}_{0,j}(m,n+2;\beta,\alpha,z_{1,l},z_{1,l}) - c_{1,l-j-1}^*\mathcal{B}_{2,j}(m,n+2;\beta,\alpha,z_{1,l},z_{1,l}) \Big) \Big]$$
$$-\mathcal{S}(m,n+2;\beta,\alpha,z_{1,l},z_0) \Big[\mathcal{B}_{0,l}(0,2;z_0,z_0) - \mathcal{B}_{2,l}(0,2;z_0,z_0)$$
$$+ \sum_{j=0}^{l-1} \Big(c_{1,l-j-1}\mathcal{B}_{0,j}(0,2;z_0,z_0) - c_{1,l-j-1}^*\mathcal{B}_{2,j}(0,2;z_0,z_0) \Big) \Big]$$
$$+ \sum_{i=1}^{m} \Big\{ \mathcal{B}_{0,l}(m+1,n+1;\beta,\beta_i,\alpha,z_{1,l}) - \mathcal{B}_{2,l}(m+1,n+1;\beta,\beta_i,\alpha,z_{1,l})$$
$$+ \sum_{j=0}^{l-1} \Big(c_{1,l-j-1}\mathcal{B}_{0,j}(m+1,n+1;\beta,\beta_i,\alpha,z_{1,l}) - c_{1,l-j-1}^*\mathcal{B}_{2,j}(m+1,n+1;\beta,\beta_i,\alpha,z_{1,l}) \Big) \Big\}$$
$$+ \sum_{i=1}^{n} \Big\{ \mathcal{B}_{0,l}(m,n+2;\beta,\alpha,\alpha_i,z_{1,l}) - \mathcal{B}_{2,l}(m,n+2;\beta,\alpha,\alpha_i,z_{1,l})$$
$$+ \sum_{j=0}^{l-1} \Big(c_{1,l-j-1}\mathcal{B}_{0,j}(m,n+2;\beta,\alpha,\alpha_i,z_{1,l}) - c_{1,l-j-1}^*\mathcal{B}_{2,j}(m,n+2;\beta,\alpha,\alpha_i,z_{1,l}) \Big) \Big\}.$$

3. $\mathcal{B}_{2,l}(m,n;\beta,\alpha)$ *for* $l \geq 1$

From the definition (178)

$$\mathcal{B}_{2,l}(m,n;\beta,\alpha) = \left(\frac{i}{k_c}\right)^{m+n-1} \lim_{\gamma\to 0^+} \left(\frac{\gamma^{4l+m+n+1}}{\Lambda_1'(z_0)} \int_{-\infty}^{\infty} dv\, D_m(\beta,v)^*\, D_n(\alpha,v)\, \hat{h}_{2,l}(v) \right). \tag{199}$$

Using $\hat{h}_{2,l} = R_2(w_{2,l})\,\hat{I}_{2,l}$, the integral becomes

$$\int_{-\infty}^{\infty} dv\, D_m(\beta,v)^*\, D_n(\alpha,v)\, \hat{h}_{2,l}(v) = \frac{-i}{2k_c} \Bigg[\int_{-\infty}^{\infty} dv\, \frac{D_m^*\, D_n\, \hat{I}_{2,l}(v)}{v - z_{2,l}} \tag{200}$$
$$- \frac{1}{4\Lambda_2(z_{2,l})} \int_{-\infty}^{\infty} dv\, \frac{D_m^*\, D_n\, \eta(v)}{v - z_{2,l}} \int_{-\infty}^{\infty} dv'\, \frac{\hat{I}_{2,l}(v')}{v' - z_{2,l}} \Bigg],$$

and with (189) - (190) we obtain

$$\mathcal{B}_{2,l}(m,n;\beta,\alpha) = -\frac{1}{2}\left(\frac{i}{k_c}\right)^{m+n}\left(1+\frac{\delta_{m,0}\,\delta_{n,0}}{3}\right)\lim_{\gamma\to 0^+}\left(\frac{\gamma^{4l+m+n+1}}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\,\frac{D_m^*\,D_n\,\hat{I}_{2,l}(v)}{v-z_{2,l}}\right). \tag{201}$$

Let $\alpha' = (\alpha, z_{2,l})$ and define $D_{n+1}(\alpha', v) = D_n(\alpha, v)/(v - z_{2,l})$, then using (175) the limit in (201) yields:

$$\begin{aligned}
&\left(\frac{i}{k_c}\right)^{m+n}\lim_{\gamma\to 0^+}\left(\frac{\gamma^{4l+m+n+1}}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\,\frac{D_m(\beta,v)^*\,D_n(\alpha,v)\,\hat{I}_{2,l}(v)}{v-z_{2,l}}\right) = \qquad (202)\\
&-\sum_{j=0}^{l-1}\left[(2+j)b_{l-j}+jb_{l-j}^*\right]\,\mathcal{B}_{2,j}(m,n+1;\beta,\alpha')\\
&+\sum_{i=1}^{m}\Big\{\mathcal{B}_{1,l-1}(m+1,n+1;\beta,\beta_i,\alpha')-\mathcal{B}_{3,l-1}(m+1,n+1;\beta,\beta_i,\alpha')\\
&\qquad\qquad -c_{1,l-1}\mathcal{S}(m+1,n+2;\beta,\beta_i,\alpha',z_0)\\
&\qquad\qquad +\sum_{j=0}^{l-2}\left[c_{1,l-j-2}\mathcal{B}_{1,j}(m+1,n+1;\beta,\beta_i,\alpha')-c_{1,l-j-2}^*\mathcal{B}_{3,j}(m+1,n+1;\beta,\beta_i,\alpha')\right]\Big\}\\
&+\sum_{i=1}^{n+1}\Big\{\mathcal{B}_{1,l-1}(m,n+2;\beta,\alpha',\alpha_i')-\mathcal{B}_{3,l-1}(m,n+2;\beta,\alpha',\alpha_i')\\
&\qquad\qquad -c_{1,l-1}\mathcal{S}(m,n+3;\beta,\alpha',\alpha_i',z_0)\\
&\qquad\qquad +\sum_{j=0}^{l-2}\left[c_{1,l-j-2}\mathcal{B}_{1,j}(m,n+2;\beta,\alpha',\alpha_i')-c_{1,l-j-2}^*\mathcal{B}_{3,j}(m,n+2;\beta,\alpha',\alpha_i')\right]\Big\}.
\end{aligned}$$

4. $\mathcal{B}_{k,l}(m,n;\beta,\alpha)$ for $k \geq 3$, $l \geq 0$

From the definition (178)

$$\mathcal{B}_{k,l}(m,n;\beta,\alpha) = \left(\frac{i}{k_c}\right)^{m+n-1}\lim_{\gamma\to 0^+}\left(\frac{\gamma^J}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\,D_m(\beta,v)^*\,D_n(\alpha,v)\,\hat{h}_{k,l}(v)\right) \tag{203}$$

where $J = 2k + 4l - 3 + m + n$. Using $\hat{h}_{k,l} = R_k(w_{k,l})\,\hat{I}_{k,l}$

$$\begin{aligned}
\int_{-\infty}^{\infty} dv\,D_m(\beta,v)^*\,D_n(\alpha,v)\,\hat{h}_{k,l}(v) = \frac{-i}{kk_c}\Bigg[&\int_{-\infty}^{\infty} dv\,\frac{D_m^*\,D_n\,\hat{I}_{k,l}(v)}{v-z_{k,l}} \qquad (204)\\
&-\frac{1}{k^2\Lambda_k(z_{k,l})}\int_{-\infty}^{\infty} dv\,\frac{D_m^*\,D_n\,\eta(v)}{v-z_{k,l}}\int_{-\infty}^{\infty} dv'\,\frac{\hat{I}_{k,l}(v')}{v'-z_{k,l}}\Bigg],
\end{aligned}$$

and with (189) - (190) we obtain

$$\mathcal{B}_{k,l}(m,n;\beta,\alpha) = -\frac{1}{k}\left(\frac{i}{k_c}\right)^{m+n}\left(1+\frac{\delta_{m,0}\,\delta_{n,0}}{k^2-1}\right)\lim_{\gamma\to 0^+}\left(\frac{\gamma^J}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\,\frac{D_m(\beta,v)^*\,D_n(\alpha,v)\,\hat{I}_{k,l}(v)}{v-z_{k,l}}\right). \tag{205}$$

Let $\alpha' = (\alpha, z_{k,l})$ and define $D_{n+1}(\alpha', v) = D_n(\alpha, v)/(v - z_{k,l})$, then from (176) we find

$$\begin{aligned}
&\left(\frac{i}{k_c}\right)^{m+n}\lim_{\gamma\to 0^+}\left(\frac{\gamma^J}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\,\frac{D_m^*\,D_n\,\hat{I}_{k,l}(v)}{v-z_{k,l}}\right) = -\sum_{j=0}^{l-1}\left[(k+j)b_{l-j}+jb_{l-j}^*\right]\,\mathcal{B}_{k,j}(m,n+1;\beta,\alpha') \\
&+\sum_{i=1}^{m}\Big\{\mathcal{B}_{k-1,l}(m+1,n+1;\beta,\beta_i,\alpha') - \mathcal{B}_{k+1,l-1}(m+1,n+1;\beta,\beta_i,\alpha') \qquad (206)\\
&\qquad +\sum_{j=0}^{l-1}c_{1,l-j-1}\mathcal{B}_{k-1,j}(m+1,n+1;\beta,\beta_i,\alpha') - \sum_{j=0}^{l-2}c_{1,l-j-2}^*\mathcal{B}_{k+1,j}(m+1,n+1;\beta,\beta_i,\alpha')\Big\} \\
&+\sum_{i=1}^{n+1}\Big\{\mathcal{B}_{k-1,l}(m,n+2;\beta,\alpha',\alpha_i') - \mathcal{B}_{k+1,l-1}(m,n+2;\beta,\alpha',\alpha_i')\Big\} \\
&\qquad +\sum_{j=0}^{l-1}c_{1,l-j-1}\mathcal{B}_{k-1,j}(m,n+2;\beta,\alpha',\alpha_i') - \sum_{j=0}^{l-2}c_{1,l-j-2}^*\mathcal{B}_{k+1,j}(m,n+2;\beta,\alpha',\alpha_i')\Big\}.
\end{aligned}$$

5. Evaluation of $c_{k,l}(\mu_c)$

From (151) and (92), we have

$$c_{k,l}(\mu_c) = \lim_{\gamma\to 0^+}\left[\gamma^{(2k+4l-3+5\delta_{k,1})}\left(\frac{-ik/k_c}{k^2-1+\Lambda_1(z_{k,l})}\right)\int_{-\infty}^{\infty} dv\,\frac{I_{k,l}(v)}{v-z_{k,l}}\right] \tag{207}$$

for $k \geq 1$ and $l \geq 0$. Since $\Lambda_1(z_{k,l}) = \Lambda_1(z_0 + i\gamma d_{k,l}/k_c) = i\gamma d_{k,l}\Lambda_1'(z_0)/k_c + \mathcal{O}(\gamma^2)$, as $\gamma \to 0^+$ the prefactor gives

$$\left(\frac{-ik/k_c}{k^2-1+\Lambda_1(z_{k,l})}\right) = \begin{cases} -\frac{1}{\gamma}\left[\frac{1}{d_{1,l}\Lambda_1'(z_0)} + \mathcal{O}(\gamma)\right] & k=1 \\[2ex] \frac{-ik/k_c}{k^2-1} + \mathcal{O}(\gamma) & k>1 \end{cases} \tag{208}$$

where $d_{1,l} = 2(1+l)$. Thus for $k = 1$

$$c_{1,l}(\mu_c) = -\lim_{\gamma\to 0^+}\left(\frac{\gamma^{4l+3}}{d_{1,l}\,\Lambda_1'(z_0)}\int_{-\infty}^{\infty} dv\frac{\hat{I}_{1,l}(v)}{v-z_{1,l}}\right), \tag{209}$$

and the right hand side is evaluated by setting $m = n = 0$ in (198):

$$
\begin{aligned}
c_{1,l}(\mu_c) = -\left(\frac{1}{d_{1,l}}\right)\Bigg\{ & -\sum_{j=0}^{l-1}\left[(2+j)b_{l-j} + (1+j)b_{l-j}^*\right]\mathcal{B}_{1,j}(0,1;z_{1,l}) \\
& + \Bigg[\mathcal{B}_{0,l}(0,2;z_{1,l},z_{1,l}) - \mathcal{B}_{2,l}(0,2;z_{1,l},z_{1,l}) \\
& \qquad + \sum_{j=0}^{l-1}\Big(c_{1,l-j-1}\mathcal{B}_{0,j}(0,2;z_{1,l},z_{1,l}) - c_{1,l-j-1}^*\mathcal{B}_{2,j}(0,2;z_{1,l},z_{1,l})\Big)\Bigg] \\
& - \Bigg[\mathcal{B}_{0,l}(0,2;z_0,z_0) - \mathcal{B}_{2,l}(0,2;z_0,z_0) \\
& \qquad + \sum_{j=0}^{l-1}\Big(c_{1,l-j-1}\mathcal{B}_{0,j}(0,2;z_0,z_0) - c_{1,l-j-1}^*\mathcal{B}_{2,j}(0,2;z_0,z_0)\Big)\Bigg]\Bigg\}.
\end{aligned} \tag{210}
$$

In deriving (210), the relation $\mathcal{S}(0,2;z_{1,l},z_0) = 1$ from (B10) in Appendix B has been used.

The corresponding expression for $c_{k,l}$ when $k > 1$ is much simpler. Setting $m = n = 0$ in (202) and (206), yields

$$
c_{k,l}(\mu_c) = \frac{i\Lambda_1'(\omega_c/k_c + i0^+)}{k_c}\mathcal{B}_{k,l}(0,0;\beta,\alpha) \tag{211}
$$

from (207) for $k \geq 2$. Here ω_c/k_c is the phase velocity at criticality; note also that when $m = n = 0$ the arguments (β,α) in $\mathcal{B}_{k,l}(m,n;\beta,\alpha)$ are irrelevant.

C. Evaluation of the lowest order singularities b_1 and b_2

It is instructive to calculate the cubic singularity b_1 and the fifth order singularity b_2 from these integrated recursion relations. In the cubic case this merely recovers the previous result (168), but the calculation illustrates the formalism. The procedure is to apply the recursion relations until expressions involving $\mathcal{S}(m,n;\beta,\alpha)$ are obtained, then the results from Appendix B are used.

1. Calculation of b_1

For $j = 1$, the expression for b_j in (184) yields

$$
b_1(\mu_c) = \mathcal{B}_{2,0}(0,2;z_0,z_0) - \mathcal{B}_{0,0}(0,2;z_0,z_0), \tag{212}
$$

and from (187) - (188) one has

$$\mathcal{B}_{0,0}(0,2;z_0,z_0) = 2\,\mathcal{S}(1,4;z_0,z_0,z_0,z_0,z_0) \tag{213}$$

$$\mathcal{B}_{2,0}(0,2;z_0,z_0) = \mathcal{S}(0,5;z_0,z_0,z_{2,0},z_0,z_0) + \frac{1}{2}\mathcal{S}(0,5;z_0,z_0,z_{2,0},z_{2,0},z_0). \tag{214}$$

In Appendix B the functions $\mathcal{S}(m,n;\beta,\alpha)$ are evaluated; applying these results yields

$$\begin{aligned}
\mathcal{B}_{0,0}(0,2;z_0,z_0) &= 2\,\mathcal{S}(1,4;z_0,z_0,z_0,z_0,z_0) \\
&= 2\left[d(1,4;0,0,0,0,0) - d(4,1;0,0,0,0,0)e^{i\xi}\right] \\
&= 2\,d(1,4;0,0,0,0,0) \\
&= \frac{1}{4},
\end{aligned} \tag{215}$$

and it follows from (B13) that $\mathcal{B}_{2,0}(0,2;z_0,z_0) = 0$. Hence, as expected, $b_1 = -1/4$ from (212).

2. Calculation of b_2

The evaluation of b_2 proceeds similarly but is considerably more laborious; I summarize the calculation below but omit the details. For $j=2$, (184) yields

$$b_2(\mu_c) = \mathcal{B}_{2,1}(0,2;z_0,z_0) - \mathcal{B}_{0,1}(0,2;z_0,z_0) - c_{1,0}\mathcal{B}_{0,0}(0,2;z_0,z_0) \tag{216}$$

since $\mathcal{B}_{2,0}(0,2;z_0,z_0) = 0$ from (214) and (B13). Setting $l=0$ in (210) gives

$$c_{1,0} = \frac{1}{2}\left[\mathcal{B}_{0,0}(0,2;z_0,z_0) - \mathcal{B}_{0,0}(0,2;z_{1,0},z_{1,0})\right]. \tag{217}$$

With $\mathcal{B}_{0,0}(0,2;z_0,z_0) = 1/4$ from (215) and a similar evaluation yielding $\mathcal{B}_{0,0}(0,2;z_{1,0},z_{1,0}) = 1/32$, this gives $c_{1,0} = 7/4^3$ and (216) becomes

$$b_2(\mu_c) = \mathcal{B}_{2,1}(0,2;z_0,z_0) - \mathcal{B}_{0,1}(0,2;z_0,z_0) - \left(\frac{7}{4^4}\right). \tag{218}$$

Consider $\mathcal{B}_{0,1}$ in (218) first; from the $k=0$ recursion relation (195)

$$\begin{aligned}
\mathcal{B}_{0,1}(0,2;z_0,z_0) = \frac{1}{4}\Big[&\frac{1}{2}\mathcal{B}_{0,0}(0,2;z_0,z_0) + 2\left(e^{i\xi}\mathcal{B}_{1,0}(3,0;z_0,z_0,z_0)^* - \mathcal{B}_{1,0}(0,3;z_0,z_0,z_0)\right) \\
&+2\,c_{1,0}\,\mathcal{S}(1,3;z_0,z_0,z_0,z_0)\Big],
\end{aligned} \tag{219}$$

and since $c_{1,0}\,\mathcal{S}(1,3;z_0,z_0,z_0,z_0) = (7/4^3)(-1/4)$, this simplifies to

$$\mathcal{B}_{0,1}(0,2;z_0,z_0) = \frac{1}{2}\left[\frac{9}{4^4} + e^{i\xi}\mathcal{B}_{1,0}(3,0;z_0,z_0,z_0)^* - \mathcal{B}_{1,0}(0,3;z_0,z_0,z_0)\right]. \tag{220}$$

For $\mathcal{B}_{2,1}$ in (218), from (201) and (202) and using the results $\mathcal{B}_{2,0}(0,3;z_0,z_0,z_{2,1}) = 0$ and $\mathcal{S}(0,5;\beta,\alpha) = 0$, one finds

$$\begin{aligned}\mathcal{B}_{2,1}(0,2;z_0,z_0) = -\frac{1}{2}\Big[2\Big(&\mathcal{B}_{1,0}(0,4;z_0,z_0,z_{2,1},z_0) - \mathcal{B}_{3,0}(0,4;z_0,z_0,z_{2,1},z_0)\Big) \\ &+\mathcal{B}_{1,0}(0,4;z_0,z_0,z_{2,1},z_{2,1}) - \mathcal{B}_{3,0}(0,4;z_0,z_0,z_{2,1},z_{2,1})\Big].\end{aligned} \tag{221}$$

From the recursion for $k = 3$ in (205), two terms in (221) vanish: $\mathcal{B}_{3,0}(0,4;z_0,z_0,z_{2,1},z_0) = 0$ and $\mathcal{B}_{3,0}(0,4;z_0,z_0,z_{2,1},z_{2,1}) = 0$, leaving

$$\mathcal{B}_{2,1}(0,2;z_0,z_0) = -\left[\mathcal{B}_{1,0}(0,4;z_0,z_0,z_{2,1},z_0) + \frac{1}{2}\mathcal{B}_{1,0}(0,4;z_0,z_0,z_{2,1},z_{2,1})\right]. \tag{222}$$

The remaining terms in (220) and (222) involving $\mathcal{B}_{1,0}$ are more tedious to evaluate from the $k = 1$ recursion relation in (197); they have the following values

$$\mathcal{B}_{1,0}(3,0;z_0,z_0,z_0) = -\frac{15\; e^{i\xi(\mu_c)}}{2^6} \tag{223}$$

$$\mathcal{B}_{1,0}(0,3;z_0,z_0,z_0) = \frac{31}{2^8} \tag{224}$$

$$\mathcal{B}_{1,0}(0,4;z_0,z_0,z_{2,1},z_0) = -\frac{379}{(3^3)(2^8)} \tag{225}$$

$$\mathcal{B}_{1,0}(0,4;z_0,z_0,z_{2,1},z_{2,1}) = -\frac{107}{(3^3)(2^7)}. \tag{226}$$

From these results (220) and (222) give

$$\mathcal{B}_{0,1}(0,2;z_0,z_0) = -\frac{41}{2^8} \tag{227}$$

$$\mathcal{B}_{2,1}(0,2;z_0,z_0) = \frac{9}{2^7}, \tag{228}$$

and from (218)

$$b_2(\mu_c) = \frac{13}{64}. \tag{229}$$

The most striking feature of this result is that, like the cubic coefficient b_1, the fifth order coefficient does not depend on the critical equilibrium $F_0(v,\mu_c)$.

D. Dependence on μ_c: the role of $e^{i\xi}$

The recursion relations in (184), (187) - (188), (195), (197) - (198), (201) - (202), (205) - (206), and (210) determine a closed set of equations for the coefficients $\{b_j\}$ and $\{c_{1,l}\}$ and the functions $\{\mathcal{B}_{k,l}\}$ (also $\{\mathcal{C}_{k,l}\}$ determined from (182)). These relations are not independent of the critical equilibrium $F_0(v, \mu_c)$, but the dependence on the parameters μ_c is entirely through a functional dependence on the phase $\exp i\xi(\mu_c)$. This implies that the higher order coefficients $b_3, b_4, \ldots$ can only depend on $F_0(v, \mu_c)$ through their dependence on $\exp i\xi(\mu_c)$; a more precise statement is given in Theorem V.1 below.

In order to describe the iterative procedure to be followed, it is helpful to organize this collection into nested subsets $\mathcal{D}(N)$ as follows. For $N = 0$ let

$$\mathcal{D}(0) \equiv \{\mathcal{B}_{0,0}, \mathcal{B}_{2,0}\} \tag{230}$$

and for $N \geq 1$ define

$$\mathcal{D}(N) \equiv \begin{cases} \{b_j(\mu_c) \mid j = 1, \ldots, N\} \\ \\ \{c_{1,l}(\mu_c) \mid l = 0, \ldots, N-1\} \\ \\ \{\mathcal{B}_{0,l} \mid l = 0, \ldots, N\} \\ \\ \{\mathcal{B}_{1,l} \mid l = 0, \ldots, N-1\} \\ \\ \{\mathcal{B}_{k,l} \mid k = 2, \ldots k_{max}(N) \;\text{ and }\; l = 0, \ldots, (k_{max}(N) - k)\} \end{cases} \tag{231}$$

where $k_{max}(N) \equiv N + 2$. In this notation it is understood that $\mathcal{B}_{k,l}$ denotes the entire set of functions $\mathcal{B}_{k,l}(m, n; \beta, \alpha)$ for all non-negative integers (m, n). Obviously $\mathcal{D}(N-1)$ is a subset of $\mathcal{D}(N)$ and as $N \to \infty$ all coefficients and functions mentioned above are included in $\mathcal{D}(N)$.

Lemma V.3 *For $N = 1, 2, \ldots$, the recursion relations determine all elements of $\mathcal{D}(N)$ in terms of the elements of $\mathcal{D}(N-1)$, the functions $\{\mathcal{S}(m, n; \beta, \alpha)\}$, and the phase $e^{i\xi}$.*

Proof.

> This is readily verified by inspection of the recursion relations, and taking into account the identity $\mathcal{C}_{1,l}(m, n; \beta, \alpha) = (-1)^{m+n-1} \mathcal{B}_{1,l}(n, m; \alpha, \beta)^* \exp i\xi$ from (182). □

The recursion relations lead to our main result concerning the dependence of the $\gamma \to 0^+$ limit on the underlying critical equilibrium $F_0(v, \mu_c)$.

Theorem V.1 *For $0 \leq N < \infty$, the elements of $\mathcal{D}(N)$ depend on the critical parameters μ_c only through a functional dependence on the phase $e^{i\xi(\mu_c)}$. In particular, for $1 \leq j < \infty$ there exist functions $Q_j(z)$, satisfying*

$$Q_j(z)^* = Q_j(z^*), \tag{232}$$

such that

$$b_j(\mu_c) = Q_j(e^{i\xi(\mu_c)}). \tag{233}$$

Each function Q_j is universal in the sense that it is independent of k_c and $F_0(v, \mu_c)$. The other elements of $\mathcal{D}(N)$ can also be similarly expressed as functions of the phase $\exp(i\xi)$ *which satisfy* (232).

Proof.

> The proof is by induction.
>
> 1. Lemma V.1 shows that $\mathcal{S}(m, n; \beta, \alpha)$ depends on μ_c only through a functional dependence on $e^{i\xi}$ and the functions involved are simple polynomials with real coefficients. Specifically $\mathcal{S}(m, n; \beta, \alpha) = s(e^{i\xi})$ where
>
> $$s(z) = d(m, n; \beta, \alpha) + (-1)^{m+n}\, d(n, m; \alpha, \beta)\, z. \tag{234}$$

It then follows immediately from (187) - (187) for $\mathcal{B}_{0,0}$ and $\mathcal{B}_{2,0}$ that these functions can also be expressed as polynomials of $e^{i\xi}$. Since $d(m,n;\beta,\alpha)$ is real-valued, these polynomials satisfy (232). This proves the theorem for $\mathcal{D}(0)$.

2. For the induction step assume that the theorem holds for $\mathcal{D}(N-1)$ for some fixed $N \geq 1$, and consider the recursion relations which determine $\mathcal{D}(N)$ from $\mathcal{D}(N-1)$ and $\{\mathcal{S}(m,n;\beta,\alpha)\}$. Each of these relations is a sum of terms and most of these terms fit the following description: the term consists of either a single element of $\mathcal{D}(N-1)$, a product of two elements of $\mathcal{D}(N-1)$, or a product of an element of $\mathcal{D}(N-1)$ with $\mathcal{S}(m,n;\beta,\alpha)$ (for some (m,n)). In each case the term is multiplied by a real coefficient which is independent of μ_c and k_c. The exceptions to this description are discussed below. For each term covered by this description if the individual elements of $\mathcal{D}(N-1)$ appearing in the term are functions of μ_c only through $e^{i\xi}$, then the entire term has this property. In addition, if the individual functions satisfy (232), then their product will also. Thus all terms, describable in this way, will depend on μ_c only through a functional dependence on $e^{i\xi}$ and this function will satisfy (232). Hence these terms preserve for $\mathcal{D}(N)$ the functional dependence on μ_c assumed for $\mathcal{D}(N-1)$.

3. The recursion relations also contain two types of terms that differ from the above description. First, there are terms which are the complex conjugate of an element of $\mathcal{D}(N-1)$, or which are the product of an element of $\mathcal{D}(N-1)$ with the complex conjugate of a second element of $\mathcal{D}(N-1)$. In either case the coefficient of the term is real and independent of μ_c and k_c. For example in (184) one finds the term $c_{1,l}^{*}\ \mathcal{B}_{2,j-l-2}(0,2;z_0,z_0)$. Since by assumption there are functions $f_1(z)$ and $f_2(z)$ such that $c_{1,l} = f_1(e^{i\xi})$ and $\mathcal{B}_{2,j-l-2}(0,2;z_0,z_0) = f_2(e^{i\xi})$, this term leads to a functional dependence

$c^*_{1,l}\,\mathcal{B}_{2,j-l-2}(0,2;z_0,z_0) = f(e^{i\xi})$ where $f(z) \equiv f_1(1/z)\, f_2(z)$. Therefore this type of term also preserves for for $\mathcal{D}(N)$ the functional dependence on μ_c assumed for $\mathcal{D}(N-1)$.

4. The second type of exceptional term involves $\mathcal{C}_{1,l}$ and only arises in the recursion relation for $\mathcal{B}_{0,l}$ in (195). Typical examples are $\mathcal{C}_{1,l-1}(m+1,n;\beta,\beta_i,\alpha)$ and $c_{1,l-j-2}\,\mathcal{C}_{1,j}(m+1,n;\beta,\beta_i,\alpha)$. Using the identity (182), these terms can be re-expressed in terms of $\mathcal{B}_{1,l}$ as:

$$\mathcal{C}_{1,l-1}(m+1,n;\beta,\beta_i,\alpha) = (-1)^{m+n}\, e^{i\xi}\,\mathcal{B}_{1,l-1}(m+1,n;\beta,\beta_i,\alpha)^*$$
$$c_{1,l-j-2}\,\mathcal{C}_{1,j}(m+1,n;\beta,\beta_i,\alpha) = (-1)^{m+n}\, e^{i\xi}\,\mathcal{B}_{1,j}(m+1,n;\beta,\beta_i,\alpha)^*.$$

In each case the essential difference from terms already discussed is that the coefficient, $(-1)^{m+n}\,e^{i\xi}$, is complex. However since this complex coefficient is obviously a function of the phase as well, it does not change the conclusion: these terms also preserve for for $\mathcal{D}(N)$ the functional dependence on μ_c assumed for $\mathcal{D}(N-1)$. Hence if the theorem holds for $\mathcal{D}(N-1)$, then it holds $\mathcal{D}(N)$. Since the conclusion has been verified above for $\mathcal{D}(0)$, by induction the theorem holds for all N.

□

When the equilibrium has reflection symmetry $F_0(v,\mu) = F_0(-v,\mu)$ then $p(\sigma,\mu)$ is real and hence $b_j = Q_j(e^{i\xi})$ must be real also. One can check that in the case of reflection symmetry $\Lambda_1'(z_0)$ is pure imaginary and $\exp i\xi = -1$. Thus the reality condition (232) ensures that in this case $b_j = Q_j(-1)$ is real as expected.

Note that the explicit calculation of b_1 and b_2 in the previous Section reveals that the first two functions in (233) are constant:

$$Q_1(z) = -\frac{1}{4} \tag{235}$$
$$Q_2(z) = \frac{13}{64}; \tag{236}$$

thus non-trivial dependence on $\exp i\xi$ can only arise in b_j for $j \geq 3$. It is natural to wonder if the Q_j are simply constants to all orders. Although this cannot be categorically ruled out, an inspection of the recursion relations does not seem encouraging. The evaluation of (236) involves a crucial and seemingly accidental cancellation that eliminates all the terms depending on $\exp i\xi$. If such a cancellation persists at still higher order, then detecting it will require a deeper understanding of the integrated recursion relations.

E. Implications of Theorem V.1

Theorem V.1 implies that as $\gamma \to 0^+$ the dynamics represented by our amplitude expansions can depend on the critical equilibrium $F_0(v, \mu_c)$ only through a functional dependence on derivative of the dielectric function evaluated at the phase velocity of the critical linear mode ω_c/k_c,

$$\Lambda_1'(\omega_c/k_c) \equiv \lim_{\gamma\to 0^+} \Lambda_1'(z_0) = \left[\text{P.V.} \int_{-\infty}^{\infty} dv \, \frac{\partial_v \eta(v, \mu_c)}{(v - \omega_c/k_c)}\right] + i\pi \frac{\partial \eta}{\partial v}(\omega_c/k_c, \mu_c). \tag{237}$$

This may also be written in terms of the phase (183),

$$\Lambda_1'(\omega_c/k_c) = |\Lambda_1'(\omega_c/k_c)| \exp(-i\xi/2); \tag{238}$$

in fact for many features it is only the phase $\exp(i\xi)$ that matters and not the magnitude.

For example, the asymptotic form of the amplitude equation (160)

$$\frac{dr}{d\tau} = r \left\{ 1 + \sum_{j=1}^{\infty} [\text{Re}\, b_j(\mu_c)]\, r^{2j} \right\} \tag{239}$$

depends on $F_0(v, \mu_c)$ through b_j, and from (233) this can be re-expressed as

$$\frac{dr}{d\tau} = r \left\{ 1 + \sum_{j=1}^{\infty} [\text{Re}\, Q_j(e^{i\xi(\mu_c)})]\, r^{2j} \right\} \tag{240}$$

where $Q_j(z)$ itself is a universal function, i.e. independent of F_0. For the electric field in (167), the Fourier component at the wavelength of the unstable mode is

$$|k_c\, E_1(t)| = \gamma^2\, r(\tau)\, \left|1 + \Gamma_1^c r(\tau)^4\right| \tag{241}$$

where

$$\Gamma_1^c = \sum_{l=0}^{\infty} c_{1,l} r^{2l}. \tag{242}$$

Since according to Theorem V.1, the coefficients $\{c_{1,l}\}$ also depend on $F_0(v, \mu_c)$ only through a universal functional dependence on $\exp(i\xi)$, this component of $E(x,t)$ is predicted to have an asymptotic dynamics that is determined by $\exp(i\xi)$. For the other wavelengths, one finds that there is an overall factor of $\Lambda_1'(\omega_c/k_c)$ in $E_k(t)$ (cf. (211)), so that $|k_c\, E_k(t)/\Lambda_1'(\omega_c/k_c)|$ is determined by $\exp(i\xi)$.

VI. THE DISTRIBUTION FUNCTION: ASYMPTOTIC BEHAVIOR

The evolving distribution function $F(x,v,t) = F_0(v,\mu) + f^u(x,v,t)$, rewritten using (47), (51) and (53), takes the form

$$\begin{aligned} F(x,v,t) = {} & F_0(v,\mu) + \sigma h_0(v,\sigma) \\ & + \left[\rho(t)\ (\psi_c(v) + \sigma h_1(v,\sigma))\ e^{i(k_c x - \theta(t))} + \sum_{k=2}^{\infty} \rho^k h_k(v,\sigma) e^{ik(k_c x - \theta(t))} + cc\right] \end{aligned} \tag{243}$$

where the phase and amplitude variables $\rho(t) e^{-i\theta(t)} = A(t)$ have been used and $\sigma = |A|^2$. For fixed $\gamma > 0$, our analysis of the mode amplitude dynamics leads to the equations (156) - (157), but the long time evolution of $\rho(t)$ is difficult to predict since the higher order nonlinear terms are not negligible; this difficulty persists even when $\gamma \to 0^+$ as shown in (160). However *if* $\rho(t)$ asymptotically approaches a constant value as $t \to \infty$,

$$\rho(t) \to \rho_\infty, \tag{244}$$

then $\theta(t)$ must settle down to a fixed frequency $d\theta/dt \to \omega_\infty$ according to (157). In this event, the form of $F(x,v,t)$ in (243) must approach a travelling wave moving at a constant wave velocity ω_∞/k_c. Hence the limiting behavior in (244) implies the flow on the unstable manifold asymptotically approaches a travelling wave such as a Bernstein-Greene-Kruskal mode. [40] This asymptotic state is necessarily periodic in time due to the periodic boundary conditions.

The asymptotic form of $F(x,v,t)$ as $\gamma \to 0^+$ can be analyzed from (243) without needing to know the long time evolution. In terms of the rescaled amplitude $\rho(t) = \gamma^2 r(\gamma t)$, (243) becomes

$$F(x,v,t) = F_0(v,\mu) + r^2\,\gamma^4 h_0(v,\sigma) \tag{245}$$
$$+\left[r\,\left(\gamma^2\psi_c(v) + \gamma^6\,r^2 h_1(v,\sigma)\right)\,e^{i(k_c x-\theta(t))} + \sum_{k=2}^{\infty}\gamma^{2k}\,r^k h_k(v,\sigma)e^{ik(k_c x-\theta(t))} + cc\right].$$

For *fixed* $v \neq v_p$, as $\gamma \to 0^+$, this implies

$$\frac{F(x,v,t) - F_0(v,\mu)}{\gamma^2} = \left[r\,\psi_c(v)e^{i(k_c x-\theta(t))} + cc\right] + \mathcal{O}(\gamma^2); \tag{246}$$

thus away from the phase velocity the correction at the wavelength of the unstable mode is dominant and is given by the critical eigenfunction. At $v = v_p$, (245) must be examined more closely since the functions $h_k(v,\sigma)$ are singular when $\gamma = 0$.

A. Asymptotic behavior near $v = v_p$

Our main interest is in the structure of $F(x,v,t)$ near $v = v_p$ as $\gamma \to 0^+$. It is necessary to extract the singular behavior of $h_k(v_p,\sigma)$ and balance it against the explicit factors of γ in (245). The key idea, pointed out by Larsen [41], is to appropriately magnify the neighborhood of $v_p(\gamma)$ using a rescaled velocity coordinate for $\gamma > 0$

$$u \equiv \frac{k_c}{\gamma}(v - v_p(\gamma)). \tag{247}$$

The motivation for this definition is the resulting factorization of a resonant denominator:

$$\frac{1}{v - z_{k,l}} = \frac{k_c}{\gamma}\left[\frac{1}{u - i(1+d_{k,l})}\right], \tag{248}$$

in which the γ^{-1} singularity has been extracted and the remaining function of u is nonsingular as $\gamma \to 0^+$.

Consider the effect of this coordinate change on the linear eigenfunction (34). Note that since $\partial_v F_0(\omega_c/k_c, \mu_c) = 0$,

$$
\begin{aligned}
\eta(v_p+\gamma u/k_c,\mu) &= -\left(\frac{1}{k_c^2}\right)\frac{\partial F_0}{\partial v}(v_p(\gamma)+\gamma u/k_c,\mu(\gamma)) && (249)\\
&= -\gamma\left(\frac{1}{k_c^2}\right)\left[\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c,\mu_c)\left(\frac{dv_p(0)}{d\gamma}+\frac{u}{k_c}\right)\right.\\
&\qquad\left.+\frac{\partial^2 F_0}{\partial v\partial\mu}(\omega_c/k_c,\mu_c)\frac{d\mu(0)}{d\gamma}+\mathcal{O}(\gamma)\right]\\
&= -\frac{\gamma}{k_c}\left[\frac{u}{k_c^2}+\mathrm{Re}\left[\Lambda_1'(\omega_c/k_c)\right]+\mathcal{O}(\gamma)\right] && (250)
\end{aligned}
$$

where in the last step the first order solutions for $v_p(\gamma)$ and $\mu(\gamma)$ from (A13) in Appendix A have been used. This motivates the definitions

$$
\begin{aligned}
\bar\eta(u,\mu) &\equiv \left(\frac{k_c}{\gamma}\right)\eta(v_p+\gamma u/k_c,\mu), && (251)\\
\bar\psi_c(u) &\equiv \psi_c(v_p+\gamma u/k_c) && (252)
\end{aligned}
$$

so that

$$
\bar\psi_c(u) = \frac{-\bar\eta(u,\mu)}{u-i}. \qquad (253)
$$

Thus the eigenfunction has a non-singular limit as $\gamma\to 0$; from (250) - (251)

$$
\lim_{\gamma\to 0^+}\bar\psi_c(u) = \frac{[\mathrm{Re}\left[\Lambda_1'(\omega_c/k_c)\right]+u/k_c^2]}{(u-i)}. \qquad (254)
$$

The generalization of these definitions for the nonlinear theory is easily done for any function characterized by an index as defined in Section IV. Let $\mathcal{G}(v,\mu)$ have index Ind $[\mathcal{G}]$ then define $\bar{\mathcal{G}}(u,\mu)$

$$
\bar{\mathcal{G}}(u,\mu) \equiv \gamma^{1+\mathrm{Ind}\ [\mathcal{G}]}\,\mathcal{G}(v_p+\gamma u/k_c,\mu); \qquad (255)
$$

this is consistent with (252) since the eigenfunction has index -1. It is not hard to see that $\bar{\mathcal{G}}(u,\mu)$ will be non-singular as $\gamma\to 0^+$. In particular this definition accomplishes the goal of extracting the singularities in $h_{k,l}$:

$$
\bar h_{k,l}(u) \equiv \gamma^{1+\mathrm{Ind}\ [h_{k,l}]}\,h_{k,l}(v_p+\gamma u/k_c). \qquad (256)
$$

Applying this to $h_k(v,\sigma)$ gives

$$\begin{aligned} h_k(v_p + \gamma u/k_c, \sigma) &= \sum_{l=0}^{\infty} \; h_{k,l}(v_p + \gamma u/k_c)\, \sigma^l \\ &= \sum_{l=0}^{\infty} \; \gamma^{4l}\, h_{k,l}(v_p + \gamma u/k_c)\, r^{2l} \\ &= \sum_{l=0}^{\infty} \; \gamma^{4l} \left(\frac{1}{\gamma}\right)^{1+\mathrm{Ind}\;[h_{k,l}]} \bar{h}_{k,l}(u)\, r^{2l} \\ &= \left(\frac{1}{\gamma}\right)^{2k-2+4(\delta_{k,0}+\delta_{k,1})} \sum_{l=0}^{\infty} \; \bar{h}_{k,l}(u)\, r^{2l}; \end{aligned} \tag{257}$$

so if $J = 2k-2+4(\delta_{k,0}+\delta_{k,1})$ then $\gamma^J h_k(v_p+\gamma u/k_c, \sigma)$ is non-singular as $\gamma \to 0^+$. Formalize this observation in the definition

$$\bar{h}_k(u, r^2) \equiv (\gamma)^{2k-2+4(\delta_{k,0}+\delta_{k,1})} \; \left[h_k(v_p + \gamma u/k_c, \gamma^4\, r^2)\right] = \sum_{l=0}^{\infty} \; \bar{h}_{k,l}(u)\, r^{2l}. \tag{258}$$

The left hand side of (246) can now be evaluated at $v = v_p + \gamma u/k_c$ in terms of $\bar{\psi}_c(u)$ and $\bar{h}_k(u, r^2)$,

$$\frac{[F(x, v_p + \gamma u/k_c, t) - F_0(v_p + \gamma u/k_c, \mu)]}{\gamma^2} = g(x, u, t, \mu) \tag{259}$$

where

$$\begin{aligned} g(x, u, t, \mu) \equiv r^2 \bar{h}_0(u, r^2) + \Bigg[& r\; (\bar{\psi}_c(u) + r^2\; \bar{h}_1(u, r^2))\; e^{i(k_c x - \theta(t))} \\ & + \sum_{k=2}^{\infty} \; r^k\; \bar{h}_k(u, r^2) e^{ik(k_c x - \theta(t))} + cc \Bigg]. \end{aligned} \tag{260}$$

As $\gamma \to 0^+$, this yields a nonsingular expression $g(x, u, t, \mu_c)$ for the distribution function in the neighborhood of the phase velocity. In contrast to (246), here *all* wavelengths contribute to the leading correction at $\mathcal{O}(\gamma^2)$.

It is natural to consider whether $g(x, u, t, \mu_c)$ is in some sense universal, or equivalently to ask how does $g(x, u, t, \mu_c)$ depend on $F_0(v, \mu_c)$? Obviously there is a trivial dependence on the linear frequency ω through the factors $\exp ik(k_c x - \theta(t))$ which can be suppressed by considering the Fourier coefficients:

$$|g_0(u, t, \mu_c)| \; = r^2 \left|\bar{h}_0(u, r^2)\right| \tag{261}$$

$$|g_1(u, t, \mu_c)| \; = r \left|\bar{\psi}_c(u) + r^2 \bar{h}_1(u, r^2)\right| \tag{262}$$

and for $k \geq 2$

$$|g_k(u, t, \mu_c)| = r^k \left|\bar{h}_k(u, r^2)\right|; \tag{263}$$

these depend on $F_0(v, \mu_c)$ through $\bar{\psi}_c(u)$ and the functions $\bar{h}_k(u, r^2)$. At $\mu = \mu_c$, the eigenfunction (254) depends on the critical equilibrium only through the derivative of the dielectric function $\Lambda_1'(\omega_c/k_c)$, and the dependence of $\bar{h}_k(u, r^2)$ on $F_0(v, \mu_c)$ can be investigated by analyzing the series coefficients $\bar{h}_{k,l}(u)$ in (258). The explicit forms for $\bar{h}_{0,0}(u)$ and $\bar{h}_{2,0}(u)$ follow from (94) - (95)

$$\bar{h}_{0,0}(u) = -\frac{\partial}{\partial u}\left[\frac{\bar{\eta}(u, \mu)}{(u-i)\,(u+i)}\right] = \frac{\partial}{\partial u}\left[\frac{\bar{\psi}_c(u)}{(u+i)}\right] \tag{264}$$

$$\bar{h}_{2,0}(u) = \frac{1}{2}\left[\frac{\partial_u \bar{\psi}_c}{(u-i)} + \left(\frac{\gamma^2}{k_c^2}\right)\frac{\Lambda_1^{(2)}(z_0)\,\bar{\eta}(u, \mu)}{6(u-i)}\right]; \tag{265}$$

as $\gamma \to 0^+$, these expressions depend on μ_c only through $\bar{\psi}_c(u)$ and therefore depend on $F_0(v, \mu_c)$ only through $\Lambda_1'(\omega_c/k_c)$.

The remaining coefficients are determined iteratively with recursion relations that follow by making the change of variable $v = v_p + \gamma u/k_c$ in the truncated relations in (172) - (176) and allowing for the asymptotic behaviors in (151) and (152). Let

$$\bar{I}_{k,l}(u) \equiv \gamma^{\mathrm{Ind}\ [h_{k,l}]}\ [I_{k,l}(v_p + \gamma u/k_c)] = \gamma^{\mathrm{Ind}\ [h_{k,l}]}\left[\hat{I}_{k,l}(v_p + \gamma u/k_c) + \mathcal{O}(\gamma)\right], \tag{266}$$

then for $k = 0$

$$\bar{h}_{0,l}(u) = \frac{\bar{I}_{0,l}(u)}{2(1+l)} \tag{267}$$

with

$$\bar{I}_{0,l}(u) = \begin{cases} i\,\frac{\partial}{\partial u}(\bar{\psi}_c^*(u) - \bar{\psi}_c(u)) & l = 0 \\ \\ -\sum_{j=0}^{l-1}(1+j)(b_{l-j} + b_{l-j}^*)\,\bar{h}_{0,j}(u) & l > 0 \\ +i\,\frac{\partial}{\partial u}\left\{\left[\bar{h}_{1,l-1}^*(u) - c_{1,l-1}^*\bar{\psi}_c(u) + \sum_{j=0}^{l-2}\bar{h}_{1,j}^*(u)c_{1,l-j-2}\right] - cc\right\} & \\ \qquad +\mathcal{O}(\gamma) & \end{cases} \tag{268}$$

For $k \geq 1$, the general relation

$$\bar{h}_{k,l}(u) = \gamma^{1+\mathrm{Ind}\ [h_{k,l}]}\ [(R_k(w_{k,l})\, I_{k,l})\,(v_p + \gamma u/k_c)]\,; \tag{269}$$

applies, although the evaluation of the resolvent depends on whether $k = 1$ or $k > 1$. For $k = 1$

$$\gamma^{1+\mathrm{Ind}\ [h_{1,l}]}\, [(R_1(w_{1,l})\, I_{1,l})\,(v_p + \gamma u/k_c)] = \frac{\left[\bar{I}_{1,l}(u) + \frac{i\bar{\eta}(u,\mu_c)}{d_{1,l}\,\Lambda_1'(\omega_c/k_c)} \int_{-\infty}^{\infty} du'\ \frac{\bar{I}_{1,l}(u')}{u'-i(1+d_{1,l})}\right]}{i[u - i(1+d_{1,l})]} + \mathcal{O}(\gamma) \tag{270}$$

where

$$\begin{aligned}
\bar{I}_{1,l}(u) = &-\sum_{j=0}^{l-1} \left[(2+j)b_{l-j} + (1+j)b^*_{l-j}\right] \bar{h}_{1,j}(u) \\
&+ i\frac{\partial}{\partial u}\left\{\bar{h}_{0,l}(u) - \bar{h}_{2,l}(u) + \sum_{j=0}^{l-1}\left(\bar{h}_{0,j}(u)c_{1,l-j-1} - \bar{h}_{2,j}(u)c^*_{1,l-j-1}\right)\right\} \\
&+ \frac{i\bar{\psi}_c(u)}{\Lambda_1'(\omega_c/k_c)}\int_{-\infty}^{\infty}\frac{du'}{(u'-i)^2}\left[\bar{h}_{0,l}(u') - \bar{h}_{2,l}(u') + \sum_{j=0}^{l-1}\left(\bar{h}_{0,j}(u')c_{1,l-j-1} - \bar{h}_{2,j}(u')c^*_{1,l-j-1}\right)\right] \\
&+ \mathcal{O}(\gamma),
\end{aligned} \tag{271}$$

and for $k \geq 2$

$$\gamma^{1+\mathrm{Ind}\ [h_{k,l}]}\, [(R_k(w_{k,l})\, I_{k,l})\,(v_p + \gamma u/k_c)] = \frac{\left[\bar{I}_{k,l}(u)\right]}{ik[u - i(1+d_{k,l})]} + \mathcal{O}(\gamma) \tag{272}$$

where

$$\bar{I}_{2,l}(u) = \begin{cases} i\,\frac{\partial}{\partial u}\bar{\psi}_c(u) & l = 0 \\[2em] -\sum_{j=0}^{l-1}\left[(2+j)b_{l-j} + j b^*_{l-j}\right]\,\bar{h}_{2,j}(u) & l > 0 \\ \quad + i\,\frac{\partial}{\partial u}\left\{\bar{h}_{1,l-1}(u) - \bar{h}_{3,l-1}(u) + \bar{\psi}_c(u)c_{1,l-1}\right. & \\ \qquad \left. + \sum_{j=0}^{l-2}\left[\bar{h}_{1,j}(u)c_{1,l-j-2} - \bar{h}_{3,j}(u)c^*_{1,l-j-2}\right]\right\} + \mathcal{O}(\gamma) & \end{cases} \tag{273}$$

and

$$\begin{aligned}
\bar{I}_{k,l}(u) = &-\sum_{j=0}^{l-1}[(k+j)b_{l-j} + j\,b^*_{l-j}]\,\bar{h}_{k,j}(u) \\
&+ i\,\frac{\partial}{\partial u}\left\{\bar{h}_{k-1,l}(u) - \bar{h}_{k+1,l-1}(u) + \sum_{j=0}^{l-1}\bar{h}_{k-1,j}(u)c_{1,l-j-1} - \sum_{j=0}^{l-2}\bar{h}_{k+1,j}(u)c^*_{1,l-j-2}\right\} \\
&+ \mathcal{O}(\gamma)
\end{aligned} \tag{274}$$

for $k \geq 3$.

Inspection of these recursion relations leads to a simple characterization of the μ_c dependence of $g(x, u, t, \mu_c)$.

Theorem VI.1 *The Fourier components $|g_k(u, t, \mu_c)|$ of $g(x, u, t, \mu_c)$ in* (261) - (263) *depend on the critical equilibrium $F_0(v, \mu_c)$ only through a functional dependence on $\Lambda_1'(\omega_c/k_c)$ where*

$$\Lambda_1'(\omega_c/k_c) \equiv \lim_{\gamma \to 0^+} \Lambda_1'(z_0). \tag{275}$$

Proof.

It suffices to examine the dependence of $\bar{\psi}_c(u)$ and $\bar{h}_k(u, r^2)$ on $F_0(v, \mu_c)$. The μ_c dependence of $\bar{\psi}_c(u)\Big|_{\mu=\mu_c}$ is through $\mathrm{Re}\,[\Lambda_1'(\omega_c/k_c)]$ from (254), and the dependence of $\bar{h}_k(u, r^2) = \sum \bar{h}_{k,l}(u) r^{2l}$ will be inferred from the coefficients $\bar{h}_{k,l}(u)$.

1. By inspection from (264) - (265), the lowest order coefficients in the organization of Table I also depend on $F_0(v, \mu_c)$ through $\mathrm{Re}\,[\Lambda_1'(\omega_c/k_c)]$:
$$\bar{h}_{0,0}(u)\Big|_{\mu=\mu_c} = \frac{\partial}{\partial u}\left[\frac{\mathrm{Re}\,[\Lambda_1'(\omega_c/k_c)] + u/k_c^2}{u^2+1}\right] \tag{276}$$
$$\bar{h}_{2,0}(u)\Big|_{\mu=\mu_c} = \frac{1}{2(u-i)}\frac{\partial}{\partial u}\left[\frac{\mathrm{Re}\,[\Lambda_1'(\omega_c/k_c)] + u/k_c^2}{u-i}\right]. \tag{277}$$

2. The remaining coefficients $\{\bar{h}_{k,l}(u)\}$ are generated from $\{\bar{\psi}_c, \bar{h}_{0,0}, \bar{h}_{2,0}\}$ using the recursion relations (267) - (274) which depend explicitly on μ_c through $\Lambda_1'(\omega_c/k_c)$, $\bar{\eta}(u, \mu_c)$, and the coefficients $\{b_j\}$ and $\{c_{1,l}\}$. By Theorem V.1, the coefficients depend on μ_c only through the phase
$$e^{i\xi} = \frac{\Lambda_1'(\omega_c/k_c)^*}{\Lambda_1'(\omega_c/k_c)}, \tag{278}$$
and from (250) - (251) $\bar{\eta}(u, \mu_c)$ depends on μ_c through $\mathrm{Re}\,[\Lambda_1'(\omega_c/k_c)]$. Thus by induction, the property that $\{\bar{\psi}_c, \bar{h}_{0,0}, \bar{h}_{2,0}\}$ depend on μ_c only through $\Lambda_1'(\omega_c/k_c)$ is passed on to all the coefficients $\{\bar{h}_{k,l}(u)\}$.

□

VII. DISCUSSION

The approach followed in this paper is patterned on the well established methods of center manifold reduction and normal form analysis which have proved quite powerful in analyzing bifurcations in dissipative systems. However for the Vlasov equation the methods must be adapted. For a dissipative system undergoing a Hopf bifurcation, one can reduce the problem to a finite-dimensional submanifold for all μ sufficiently close to μ_c, and for $\mu > \mu_c$ this corresponds to the finite-dimensional unstable manifold associated with the equilibrium. By contrast, in the Vlasov instability there is no analogous possiblility of reducing to a finite-dimensional submanifold in a full neighborhood of $\mu = \mu_c$ since the critical eigenvalues first appear at $\mu = \mu_c$ embedded in the continuum and then emerge for $\mu > \mu_c$. It is only on one side of criticality ($\mu > \mu_c$) that the linear spectrum indicates finite-dimensional submanifolds should exist and offer a useful parallel to the dissipative case.

The center manifold and normal form analysis for Hopf bifurcation can be rephrased in terms of the unstable manifold by deriving the dynamics on the unstable manifold and then taking the $\gamma \to 0^+$ limit of the resulting vector field to obtain normal form equations for the amplitude $A(t)$. As pointed out in the Introduction, the scaling $A(t) = \sqrt{\gamma}\, r(\gamma t) e^{-i\theta(t)}$ then yields asymptotic equations [42]

$$\frac{dr}{d\tau} = r R_H(r,\mu) \tag{279}$$

$$\frac{d\theta}{dt} = \omega + \mathcal{O}(\gamma^2) \tag{280}$$

where

$$R_H(r,\mu) = 1 + [a_1(\mu_c) + \mathcal{O}(\gamma)]r^2 + \mathcal{O}(\gamma^2). \tag{281}$$

This result reveals the correct scaling $\sqrt{\gamma}$ for the nonlinear oscillation and shows that the expansion (281) for the amplitude dynamics can be truncated after the lowest order nonlinear term. After truncation, the remaining dependence on the underlying critical equilibrium is expressed entirely through the lowest order nonlinear coefficient $a_1(\mu_c)$ which is a calculable

function of μ_c. This truncation also allows the time-asymptotic behavior to be determined since the solutions of $R_H = 0$ are then easily found.

Our application of this same procedure to the instability of an electrostatic wave in a collsionless plasma leads to equations like (279) - (280) with several qualitative differences. First, the scaling of the mode amplitude required to obtain nonsingular asymptotic equations is quite different: $A(t) = \gamma^2\, r(\gamma t) e^{-i\theta(t)}$ where now

$$\frac{dr}{d\tau} = rR(r,\mu) \tag{282}$$

$$\frac{d\theta}{dt} = \omega + \mathcal{O}(\gamma) \tag{283}$$

with

$$R(r,\mu) = 1 + \sum_{j=1}^{\infty} [\mathrm{Re}\ Q_j(e^{i\xi(\mu_c)})\, r^{2j} + \mathcal{O}(\gamma)]. \tag{284}$$

Secondly, the expansion for the amplitude equation (284) cannot be truncated. This means that the solutions to $R(r,\mu_c) = 0$ are not readily found and consequently (282) makes no obvious prediction for the long time behavior. However despite the greater complexity of $R(r,\mu_c)$, the dependence on μ_c can be characterized as arising only through a functional dependence on the phase $e^{i\xi(\mu_c)}$ which is in turn determined from the derivative of the dielectric function.

This characterization of $R(r,\mu_c)$ (and similar statements for the electric field and distribution function) makes testable predictions about instabilities driven by physically very different distributions. If $e^{i\xi(\mu_c)}$ is fixed, then any variations in densities or temperatures characterizing $F_0(v,\mu_c)$ do not affect the evolution of $r(\tau)$. For example, a beam-plasma instability (complex λ) and a two-stream instability (real λ), compared at a common value of $e^{i\xi}$, have identical amplitude equations up to $\mathcal{O}(\gamma)$ corrections. Examination of this prediction through numerical solution of the Vlasov equation will be undertaken in a future paper.

A final important difference with the dissipative case concerns the generality of the analysis. In standard center manifold reduction, the local attractivity of the submanifold

ensures that the time-asymptotic behavior of any initial condition near the equilibrium can be reliably predicted from the evolution on the submanifold. For Vlasov, or more generally in a Hamiltonian bifurcation, when there are neutral modes in addition to the critical modes, then there may be little or no correlation between the evolution of an initial condition on the unstable manifold and the evolution of an arbitrary initial condition. Nevertheless numerical studies of the one mode instablility observe the trapping scaling in the saturation amplitude of the electric field as a quite robust phenomenon; there is no evidence that the initial condition must be carefully chosen. This indicates that some features of the evolution on the unstable manifold have a wider validity, but it is not clear what selects these features. One appealing conjecture is that for initial conditions near F_0, but not on the unstable manifold, the electric field $E(x,t)$ evolves asymptotically towards the electric field $E^u(x,t)$ associated with a solution on the unstable manifold. This could occur without requiring a corresponding asymptotic behavior in $F(x,v,t)$ and would suffice to explain the robustness of the trapping scaling. This picture seems difficult to investigate analytically but can be tested through numerical experiments.

A basic feature of the dynamics expected from an autonomous one-dimensional flow such as $dr/d\tau = R(r,\mu_c)$ is the absence of oscillations, provided R is differentiable in r (or at least Lipschitz continuous). This absence of oscillations implies that if $r(\tau)$ approaches an asymptotic limit $r(\tau) \to r_\infty$as $\tau \to \infty$, then the approach will be monotonic from below. This is indeed found in Hopf bifurcation where $R_H(r,\mu_c)$ is simply a quadratic polynomial in r. However in numerical studies of the one mode instability such monotonic relaxation in the time-asymptotic regime is not observed, rather one finds the familiar trapping oscillations in the electric field. [3]

The explanation of trapping oscillations within the setting of unstable manifold dynamics may be related to the survival of an infinite sum of terms in (284) as $\gamma \to 0^+$. The fact that higher order terms in r are not higher order in γ is a marked contrast with the much simpler limit found in the Hopf normal form (281), and is directly related to the fact that the critical eigenvalues merge with the continuous spectrum as $\gamma \to 0^+$. In exactly solvable models,

where neutral modes also introduce singularities into the amplitude equation, I have shown that the unstable manifold can develop a spiral structure which persists as $\gamma \to 0^+$ and this spiral allows the flow to approach an asymptotic limit through a decaying oscillation. [43] This spiral structure is illustrated in Fig. 4.

When such a spiral is present, then describing the dynamics on the manifold via a mapping H from the unstable subspace (45) yields a vector field on E^u with branch point singularities at the points where the flow moves from one branch of the spiral to the next. In the solvable examples, the fact that $R(r, \mu_c)$ has a branch point within the domain of flow implies that the higher order terms in the expansion of $R(r, \mu_c)$ remain essential to the dynamics even as $\gamma \to 0^+$. Supposing that the amplitude dynamics for the one mode problem has a similar structure, then $R(r, \mu_c)$ would have a branch point at r_b, the turning point of the spiral. As the mode grows, the increase of $r(\tau)$ to r_b would signal the onset of trapping oscillations with the passage of the trajectory to the next branch of the unstable manifold.

Note that a trajectory will reach such a branch point node in *finite* time, unlike the more familiar situation of a node where the vector field is differentiable and the approach time is infinite. In addition, the loss of smoothness at $r = r_b$ introduces the lack of uniqueness needed by the solution to pass through the branch point. Finally it should be clear that such a spiral structure would present a significant obstacle to using the power series (284) to determine the time-asymptotic amplitude r_∞.

Instabilities in other systems, including ideal shear flows [44–46], solitary waves [47–49], bubble clouds [50], and globally-coupled populations of oscillators [51,52], also exhibit key features of this problem, most notably that the unstable modes correspond to eigenvalues emerging from a neutral continuum at onset. In the case of ideal shear flows similar singularities arise in the amplitude equations for the unstable modes (γ^{-3} at cubic order) [46]; by contrast, in globally coupled phase models for the onset of synchronized behavior in a population of oscillators the critical eigenvalues emerge from the continuum at the onset of

instability but the amplitude equations are nonsingular and $\sqrt{\gamma}$ scaling is found (at least in the best understood case of a real eigenvalue). [37,51,53] This difference in the nonlinear behavior seems noteworthy since the linear dynamics of the oscillator model is qualitatively similar to Vlasov although apparently lacking a Hamiltonian structure. [52] In the models of solitary waves and bubble clouds, the scaling behavior in the weakly unstable regime has not been investigated. It would be interesting to determine if singularities arise in the amplitude equations for the unstable modes in these problems with corresponding implications for the scaling behavior of the nonlinear states.

The study of these novel bifurcations from a unified viewpoint is just beginning, and it is not yet clear how to abstract the essential features required to produce singular expansions and unusual scaling behavior.

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APPENDIX A: PARAMETRIZED ROOTS OF THE DIELECTRIC FUNCTION

For $\gamma \geq 0$, the roots of the dielectric function determine the eigenvalues of the unstable modes. From (29) this requires the phase velocity $v_p(\gamma)$ and parameter $\mu(\gamma)$ to satisfy

$$\Lambda_1(z_0(\gamma), \mu(\gamma)) = 1 - \frac{1}{k_c^2} I(\gamma) = 0 \tag{A1}$$

where

$$I(\gamma) \equiv \int_{-\infty}^{\infty} \frac{dv\, \partial_v F_0(v, \mu(\gamma))}{v - v_p(\gamma) - i\gamma/k_c}. \tag{A2}$$

For simplicity, μ is taken to be a single parameter.

From (A1) these functions can be calculated perturbatively for small γ; let

$$v_p(\gamma) = \frac{\omega_c}{k_c} + v_1\,\gamma + \mathcal{O}(\gamma^2) \tag{A3}$$

$$\mu(\gamma) = \mu_c + \mu_1\,\gamma + \mathcal{O}(\gamma^2), \tag{A4}$$

and also expand $I(\gamma)$:

$$I(\gamma) = I(0) + \gamma \frac{dI}{d\gamma}(0) + \mathcal{O}(\gamma^2). \tag{A5}$$

Then (A1) implies

$$I(0) = k_c^2 \tag{A6}$$

$$\frac{dI}{d\gamma}(0) = 0 \tag{A7}$$

and so forth; the coefficients (v_1, μ_1) are determined by (A7). Define $v = v_p(\gamma) + u/k_c$ and take u as the variable of integration in I,

$$I(\gamma) = \int_{-\infty}^{\infty} \frac{du}{u - i\gamma} \frac{\partial F_0}{\partial v}(v_p(\gamma) + u, \mu(\gamma)), \tag{A8}$$

then

$$\begin{aligned} \frac{dI}{d\gamma}(0) = \mathrm{P.V.} \int_{-\infty}^{\infty} \frac{du}{u} & \left[\left(\frac{i}{k_c} + v_1 \right) \frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c + u/k_c, \mu_c) + \mu_1 \frac{\partial^2 F_0}{\partial v \partial \mu}(\omega_c/k_c + u/k_c, \mu_c) \right] \\ & + i\pi \left[\left(\frac{i}{k_c} + v_1 \right) \frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c, \mu_c) + \mu_1 \frac{\partial^2 F_0}{\partial v \partial \mu}(\omega_c/k_c, \mu_c) \right]. \end{aligned} \tag{A9}$$

Since the real and imaginary parts must vanish separately in (A7), there are two equations for v_1 and μ_1:

$$\frac{1}{k_c}\left[\text{P.V.}\int_{-\infty}^{\infty}\frac{du}{u}\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c+u/k_c,\mu_c)\right]$$
$$+\pi\left[v_1\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c,\mu_c)+\mu_1\frac{\partial^2 F_0}{\partial v\partial\mu}(\omega_c/k_c,\mu_c)\right]=0 \tag{A10}$$
$$\text{P.V.}\int_{-\infty}^{\infty}\frac{du}{u}\left[v_1\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c+u/k_c,\mu_c)+\mu_1\frac{\partial^2 F_0}{\partial v\partial\mu}(\omega_c/k_c+u/k_c,\mu_c)\right]$$
$$-\frac{\pi}{k_c}\left[\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c,\mu_c)\right]=0 \tag{A11}$$

With

$$\Lambda_1'(\omega_c/k_c)=-\frac{1}{k_c^2}\left\{\left[\text{P.V.}\int_{-\infty}^{\infty},\frac{du}{u}\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c+u/k_c,\mu_c)\right]+i\pi\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c,\mu_c)\right\} \tag{A12}$$

from (237), these equations may be rewritten more simply as

$$v_1\left[\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c,\mu_c)\right]+\mu_1\left[\frac{\partial^2 F_0}{\partial v\partial\mu}(\omega_c/k_c,\mu_c)\right]=k_c\text{Re}\,[\Lambda_1'(\omega_c/k_c)] \tag{A13}$$
$$v_1\left[\text{P.V.}\int_{-\infty}^{\infty}\frac{du}{u}\frac{\partial^2 F_0}{\partial v^2}(\omega_c/k_c+u/k_c,\mu_c)\right]$$
$$+\mu_1\left[\text{P.V.}\int_{-\infty}^{\infty}\frac{du}{u}\frac{\partial^2 F_0}{\partial v\partial\mu}(\omega_c/k_c+u/k_c,\mu_c)\right]=-k_c\text{Im}\,[\Lambda_1'(\omega_c/k_c)]. \tag{A14}$$

APPENDIX B: EVALUATION OF SINGULAR INTEGRALS

Let $D_0\equiv 1$ and for $n>0$ define

$$D_n(\alpha,v)\equiv\frac{1}{(v-\alpha_1)(v-\alpha_2)\cdots(v-\alpha_n)} \tag{B1}$$

where $\alpha\equiv(\alpha_1,\ldots,\alpha_n)$. For non-negative integers (m,n) such that $m+n\geq 2$, we consider limits of the following type (c.f. (177)):

$$\mathcal{S}(m,n;\beta,\alpha)\equiv\left(\frac{i}{k_c}\right)^{m+n-2}\lim_{\gamma\to 0^+}\left(\frac{\gamma^{m+n-2}}{\Lambda_1'(z_0)}\int_{-\infty}^{\infty}dv\,D_m(\beta,v)^*\,D_n(\alpha,v)\,\eta(v,\mu)\right), \tag{B2}$$

where the poles of the integrand

$$\alpha_j=z_0+i\gamma\nu_j/k_c \qquad j=1,\ldots,n \tag{B3}$$

$$\beta_j^*=z_0^*-i\gamma\zeta_j/k_c \qquad j=1,\ldots,m \tag{B4}$$

lie along the vertical line Re $v = v_p$. The non-negative constants $\nu_j \geq 0$ and $\zeta_j \geq 0$ are assumed to be independent of F_0 for all j; in particular they are independent of γ. The limit $\mathcal{S}(m, n; \beta, \alpha)$ does depend on F_0 as determined below.

Several general relations can be noted immediately; first interchanging the order of the arguments gives the simple identity

$$\mathcal{S}(n, m; \alpha, \beta) = (-1)^{m+n}\, e^{i\xi(\mu_c)}\, \mathcal{S}(m, n; \beta, \alpha)^* \tag{B5}$$

where

$$e^{i\xi(\mu_c)} = \lim_{\gamma \to 0^+} \left(\frac{\Lambda_1'(z_0)^*}{\Lambda_1'(z_0)} \right) \tag{B6}$$

is the phase defined previously in (183). Thus it is sufficient to evaluate (B2) for $m \leq n$. Secondly, by expanding the denominator with partial fractions, the limit $\mathcal{S}(m, n; \beta, \alpha)$ can be expressed in terms of the limits for $m + n - 1$. If $m > 1$ then

$$\mathcal{S}(m, n; \beta, \alpha) = \frac{[\mathcal{S}(m-1, n; \beta'', \alpha) - \mathcal{S}(m-1, n; \beta', \alpha)]}{\zeta_{m-1} - \zeta_m} \tag{B7}$$

can be used, if $n > 1$ then

$$\mathcal{S}(m, n; \beta, \alpha) = \frac{[\mathcal{S}(m, n-1; \beta, \alpha'') - \mathcal{S}(m, n-1; \beta, \alpha')]}{\nu_n - \nu_{n-1}} \tag{B8}$$

applies, and if $m > 1$ and $n > 1$ then

$$\mathcal{S}(m, n; \beta, \alpha) = \frac{[\mathcal{S}(m-1, n; \beta', \alpha) - \mathcal{S}(m, n-1; \beta, \alpha')]}{2 + \nu_n + \zeta_m} \tag{B9}$$

can be used. In these recursion relations, the primed arguments are defined by $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$, $\alpha'' = (\alpha_1, \ldots, \alpha_{n-2}, \alpha_n)$, $\beta' = (\beta_1, \ldots, \beta_{m-1})$, and $\beta'' = (\beta_1, \ldots, \beta_{m-2}, \beta_m)$.

1. Proof of Lemma V.1

These general properties allow $\mathcal{S}(m, n; \beta, \alpha)$ to be calculated recursively from results for $m + n = 2, 3$. For $m + n = 2$ the integral is straightforward to evaluate and the limits are given by

$$\mathcal{S}(0,2;\alpha_1,\alpha_2) = 1 \tag{B10}$$

$$\mathcal{S}(2,0;\beta_1,\beta_2) = e^{i\xi} \tag{B11}$$

$$\mathcal{S}(1,1;\beta_1,\alpha_1) = \frac{\nu_1 + \zeta_1\, e^{i\xi}}{2+\nu_1+\zeta_1}. \tag{B12}$$

For $m+n \geq 2$, when either $m=0$ or $n=0$, the integral is nonsingular as $\gamma \to 0^+$, so the results in (B10) and (B11) generalize easily

$$\mathcal{S}(0,n;\alpha) = \delta_{n,2}. \tag{B13}$$

$$\mathcal{S}(m,0;\beta) = \delta_{m,2}\, e^{i\xi}. \tag{B14}$$

A similar evaluation for $m+n=3$ gives two more limits

$$\mathcal{S}(1,2;\beta_1,\alpha_1,\alpha_2) = \frac{(2+\zeta_1) - \zeta_1\, e^{i\xi}}{(2+\nu_1+\zeta_1)(2+\nu_2+\zeta_1)} \tag{B15}$$

$$\mathcal{S}(2,1;\beta_1,\beta_2,\alpha_1) = \frac{\nu_1 - (2+\nu_1)\, e^{i\xi}}{(2+\nu_1+\zeta_1)(2+\nu_1+\zeta_2)}. \tag{B16}$$

By inspection of (B10) - (B16), the results for $m+n=2,3$ have the following form:

$$\mathcal{S}(m,n;\beta,\alpha) = d(m,n;\zeta,\nu) + (-1)^{m+n} d(n,m;\nu,\zeta)\, e^{i\xi} \tag{B17}$$

where $\zeta = (\zeta_1,\ldots,\zeta_m)$, $\nu = (\nu_1,\ldots,\nu_n)$, and the functions $d(m,n;\zeta,\nu)$ are given by

$$d(0,2;\nu_1,\nu_2) = 1 \tag{B18}$$

$$d(2,0;\zeta_1,\zeta_2) = 0 \tag{B19}$$

$$d(1,1;\zeta_1,\nu_1) = \frac{\nu_1}{2+\nu_1+\zeta_1} \tag{B20}$$

for $m+n=2$ and

$$d(1,2;\zeta_1,\nu_1,\nu_2) = \frac{2+\zeta_1}{(2+\nu_1+\zeta_1)(2+\nu_2+\zeta_1)} \tag{B21}$$

$$d(2,1;\zeta_1,\zeta_2,\nu_1) = \frac{\nu_1}{(2+\nu_1+\zeta_1)(2+\nu_1+\zeta_2)} \tag{B22}$$

for $m+n=3$.

The representation in (B17) in fact holds for all $m+n \geq 2$. From (B13) and (B14)

$$d(0, n; \nu) = \delta_{n,2} \tag{B23}$$

$$d(m, 0; \zeta) = \delta_{m,2}. \tag{B24}$$

More generally, when m and n are both non-zero, inserting (B17) into (B7) - (B9) yields the corresponding recursion relations for $d(m, n; \zeta, \nu)$:

$$d(m, n; \zeta, \nu) = \frac{d(m-1, n; \zeta', \nu) - d(m, n-1; \zeta, \nu')}{2 + \nu_n + \zeta_m} \tag{B25}$$

$$d(m, n; \zeta, \nu) = \frac{d(m, n-1; \zeta, \nu'') - d(m, n-1; \zeta, \nu')}{\nu_n - \nu_{n-1}} \tag{B26}$$

$$d(m, n; \zeta, \nu) = \frac{d(m-1, n; \zeta'', \nu) - d(m-1, n; \zeta', \nu)}{\zeta_{m-1} - \zeta_m} \tag{B27}$$

where $\zeta' = (\zeta_1, \ldots, \zeta_{m-1})$, $\nu' = (\nu_1, \ldots, \nu_{n-1})$, $\zeta'' = (\zeta_1, \ldots, \zeta_{m-2}, \zeta_m)$, and $\nu'' = (\nu_1, \ldots, \nu_{n-2}, \nu_n)$. In (B26) and (B27) there is no singularity when the denominator vanishes because the numerator also vanishes; more convenient forms for these recursions are given below.

Thus the recursive evaluation of $\mathcal{S}(m, n; \beta, \alpha)$ reduces to the recursive evaluation of $d(m, n; \zeta, \nu)$. Note that the functions $d(m, n; \zeta, \nu)$ are real-valued and universal in the sense of being independent of the equilibrium $F_0(v, \mu_c)$. The limit $\mathcal{S}(m, n; \beta, \alpha)$ depends on $F_0(v, \mu_c)$ only through the phase $e^{i\xi}$. This completes the proof of Lemma V.1.

2. Evaluation of b_2

In the remainder of this Appendix, some results for the functions $d(m, n; \zeta, \nu)$ that are useful in the calculation of b_2 are briefly summarized. The case when $m = 0$ or $n = 0$ is trivial, and from (B21) and the recursion relation in (B25), it is easy to show that for $n \geq 2$

$$d(1, n; \zeta_1, \nu) = \frac{(-1)^n (2 + \zeta_1)}{\prod_{i=1}^{n} (2 + \nu_i + \zeta_1)}. \tag{B28}$$

Thus the functions are explicitly known unless both m and n are greater than 1 so that $m + n \geq 4$.

Henceforth assume that both m and n are non-zero and that $m+n \geq 3$. In (B21) - (B22), note that $d(2,1;\zeta_1,\zeta_2,\nu_1) = d(1,2;\nu_1-2,2+\zeta_1,2+\zeta_2)$; more generally such a relation also holds for $m+n>3$:

$$d(m,n;\zeta_1,\ldots,\zeta_m,\nu_1,\ldots,\nu_n) = (-1)^{m+n-1} d(n,m;\nu_1-2,\ldots,\nu_n-2,2+\zeta_1,\ldots,2+\zeta_m). \tag{B29}$$

This identity is readily verified by induction from the recursion relations for $d(m,n;\zeta,\nu)$ in (B25) - (B27). Hence it is sufficient to calculate $d(m,n;\zeta,\nu)$ for $m \leq n$.

From the definitions it is clear that $d(m,n;\zeta,\nu)$ must be symmetric under interchange of the m arguments $(\zeta_1,\ldots,\zeta_m)$ and also under interchange of the n arguments $(\nu_1,\ldots,\nu_n)$. It is convenient to have a notation that makes this more explicit and which is manifestly nonsingular when some of these arguments coincide and the denominators in (B26) and (B27) vanish. For $m \geq 1$ and $n \geq 1$, define $N(m,n;\zeta,\nu)$ by the formula

$$d(m,n;\zeta,\nu) = \frac{N(m,n;\zeta,\nu)}{\prod_{i=1}^{n}\prod_{j=1}^{m}(2+\nu_i+\zeta_j)}; \tag{B30}$$

so that

$$N(1,1;\zeta_1,\nu_1) = \nu_1 \tag{B31}$$

$$N(1,2;\zeta_1,\nu_1,\nu_2) = 2+\zeta_1 \tag{B32}$$

$$N(2,1;\zeta_1,\zeta_2,\nu_1) = \nu_1. \tag{B33}$$

From (B25) the corresponding recursion relation for $N(m,n;\zeta,\nu)$ when m and n are greater than 1 and $m+n \geq 4$ is

$$N(m,n;\zeta,\nu) = N(m-1,n;\zeta',\nu)\left[\prod_{i=1}^{n-1}(2+\nu_i+\zeta_m)\right] - N(m,n-1;\zeta,\nu')\left[\prod_{j=1}^{m-1}(2+\nu_n+\zeta_j)\right], \tag{B34}$$

and from (B29) we have the identity

$$N(m,n;\zeta_1,\ldots,\zeta_m,\nu_1,\ldots,\nu_n) = (-1)^{m+n-1} N(n,m;\nu_1-2,\ldots,\nu_n-2,2+\zeta_1,\ldots,2+\zeta_m) \tag{B35}$$

when $m + n \geq 3$.

Clearly the functions $N(m, n; \zeta, \nu)$ are also symmetric under interchange of the m arguments $(\zeta_1, \ldots, \zeta_m)$ and also under interchange of the n arguments $(\nu_1, \ldots, \nu_n)$ but this is not manifest in (B34). Some functions $N(m, n; \zeta, \nu)$, useful in the evaluation of b_2, are listed below showing explicitly the interchange symmetry; in these formulas the notation P_n corresponds to the symmetric polynomials defined by $P_1(x, y) \equiv x + y$, $P_2(x, y) \equiv x^2 + xy + y^2$, $P_3(x, y) \equiv x^3 + x^2y + xy^2 + y^3$, and $P_4(x, y) \equiv x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

$$N(2,2;\zeta,\nu) = -\nu_1\nu_2 + (2+\zeta_1)(2+\zeta_2) \tag{B36}$$

$$N(2,3;\zeta,\nu) = \nu_1\nu_2\nu_3 - (2+\zeta_1)(2+\zeta_2)\Big\{P_1(2+\zeta_1, 2+\zeta_2) + \nu_1 + \nu_2 + \nu_3\Big\} \tag{B37}$$

$$\begin{aligned} N(2,4;\zeta,\nu) = -\nu_1\nu_2\nu_3\nu_4 + (2+\zeta_1)(2+\zeta_2)\Big\{&P_2(2+\zeta_1, 2+\zeta_2) + P_1(2+\zeta_1, 2+\zeta_2)\Big[\nu_1+\nu_2+\nu_3+\nu_4\Big] \\ &+\nu_1(\nu_2+\nu_3+\nu_4) + \nu_2(\nu_3+\nu_4) + \nu_3\nu_4\Big\} \end{aligned} \tag{B38}$$

$$\begin{aligned} N(2,5;\zeta,\nu) = \nu_1\nu_2\nu_3\nu_4\nu_5 - (2+\zeta_1)(2+\zeta_2)\Big\{&P_3(2+\zeta_1, 2+\zeta_2) \\ &+P_2(2+\zeta_1, 2+\zeta_2)\Big[\nu_1+\nu_2+\nu_3+\nu_4+\nu_5\Big] \\ &+P_1(2+\zeta_1, 2+\zeta_2)\Big[\nu_1(\nu_2+\nu_3+\nu_4+\nu_5) + \nu_2(\nu_3+\nu_4+\nu_5) \\ &\qquad +\nu_3(\nu_4+\nu_5) + \nu_4\nu_5\Big] \\ &+\nu_1\Big[\nu_2(\nu_3+\nu_4+\nu_5) + \nu_3(\nu_4+\nu_5) + \nu_4\nu_5)\Big] \\ &\qquad +\nu_2\Big[\nu_3(\nu_4+\nu_5) + \nu_4\nu_5\Big] + \nu_3\nu_4\nu_5\Big\} \end{aligned} \tag{B39}$$

and

$$\begin{aligned} N(3,3;\zeta,\nu) = -(2+\zeta_1)(2+\zeta_2)(2+\zeta_3)\Big\{&(2+\zeta_1)(2+\zeta_2+2+\zeta_3) + (2+\zeta_2)(2+\zeta_3) \\ &+(6+\zeta_1+\zeta_2+\zeta_3)(\nu_1+\nu_2+\nu_3) \\ &+\nu_1^2+\nu_2^2+\nu_3^2+\nu_1(\nu_2+\nu_3)+\nu_2\nu_3\Big\} \\ +\nu_1\nu_2\nu_3\Big\{\nu_1(\nu_2+\nu_3) + \nu_2\nu_3 &+ (6+\zeta_1+\zeta_2+\zeta_3)(\nu_1+\nu_2+\nu_3) + (2+\zeta_1)^2 \\ +(2+\zeta_2)^2 + (2+\zeta_3)^2 &+ (2+\zeta_1)(2+\zeta_2+2+\zeta_3) + (2+\zeta_2)(2+\zeta_3)\Big\} \end{aligned} \tag{B40}$$

$$
\begin{aligned}
N(3,4;\zeta,\nu) = (2+\zeta_1)(2+\zeta_2)(2+\zeta_3) \Big\{ & (2+\zeta_1)^2 \left[(2+\zeta_2)^2 + (2+\zeta_3)^2\right] \\
& + (2+\zeta_2)^2(2+\zeta_3)^2 + (2+\zeta_1)(2+\zeta_2)(2+\zeta_3)[6+\zeta_1+\zeta_2+\zeta_3] \\
& + (\nu_1+\nu_2+\nu_3+\nu_4)\Big[(2+\zeta_1)^2(4+\zeta_2+\zeta_3) + (2+\zeta_2)^2(4+\zeta_1+\zeta_3) \\
& \qquad + (2+\zeta_3)^2(4+\zeta_1+\zeta_2) + 2(2+\zeta_1)(2+\zeta_2)(2+\zeta_3)\Big] \\
& + (\nu_1^2+\nu_2^2+\nu_3^2+\nu_4^2)\left[(2+\zeta_1)(2+\zeta_2+2+\zeta_3) + (2+\zeta_2)(2+\zeta_3)\right] \\
& + [\nu_1(\nu_2+\nu_3+\nu_4) + \nu_2(\nu_3+\nu_4) + \nu_3\nu_4](6+\zeta_1+\zeta_2+\zeta_3)^2 \\
& + (6+\zeta_1+\zeta_2+\zeta_3)\Big[\nu_1^2(\nu_2+\nu_3+\nu_4) + \nu_2^2(\nu_1+\nu_3+\nu_4) + \nu_3^2(\nu_1+\nu_2+\nu_4) \\
& \qquad + \nu_4^2(\nu_1+\nu_2+\nu_3) + 2(\nu_1\nu_2\nu_3 + \nu_1\nu_2\nu_4 + \nu_1\nu_3\nu_4 + \nu_2\nu_3\nu_4)\Big] \\
& + \nu_1^2(\nu_2^2+\nu_3^2+\nu_4^2) + \nu_2^2(\nu_3^2+\nu_4^2) + \nu_3^2\nu_4^2 + \nu_1^2(\nu_2\nu_3+\nu_2\nu_4+\nu_3\nu_4) \\
& + \nu_2^2(\nu_1\nu_3+\nu_1\nu_4+\nu_3\nu_4) + \nu_3^2(\nu_1\nu_2+\nu_1\nu_4+\nu_2\nu_4) \\
& + \nu_4^2(\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3) + 3\nu_1\nu_2\nu_3\nu_4 \Big\} \\
- \nu_1\nu_2\nu_3\nu_4 \Big\{ & (2+\zeta_1)^3 + (2+\zeta_2)^3 + (2+\zeta_3)^3 + (2+\zeta_1)^2(4+\zeta_2+\zeta_3) \\
& + (2+\zeta_2)^2(4+\zeta_1+\zeta_3) + (2+\zeta_3)^2(4+\zeta_1+\zeta_2) + 2(2+\zeta_1)(2+\zeta_2)(2+\zeta_3) \\
& + (\nu_1+\nu_2+\nu_3+\nu_4)\Big[(2+\zeta_1)^2 + (2+\zeta_2)^2 + (2+\zeta_3)^2 \\
& \qquad + (2+\zeta_1)(4+\zeta_2+\zeta_3) + (2+\zeta_2)(2+\zeta_3)\Big] \\
& + (6+\zeta_1+\zeta_2+\zeta_3)\left[\nu_1(\nu_2+\nu_3+\nu_4) + \nu_2(\nu_3+\nu_4) + \nu_3\nu_4\right] \\
& + \nu_1(\nu_2\nu_3+\nu_2\nu_4+\nu_3\nu_4) + \nu_2\nu_3\nu_4 \Big\}
\end{aligned} \tag{B41}
$$

REFERENCES

[1] E. Frieman, S. Bodner, and P. Rutherford, Some new results on the quasi-linear theory of plasma instabilities, Phys. Fl. **6** 1298 (1963).

[2] D.E. Baldwin, Perturbation method for waves in a slowly varying plasma, Phys. Fl. **7** 782 (1964).

[3] W.E. Drummond, J.H. Malmberg, T.M. O'Neil and J.R. Thompson, Nonlinear development of the beam-plasma instability, Phys. Fl. **13** 2422 (1970).

[4] I.N. Onischenko, A.R. Linetskii, N.G. Matsiborko, V.D. Shapiro and V.I. Shevchenko, Contribution to the nonlinear theory of excitation of a monochromatic plasma wave by an electron beam, JETP Lett. **12** 281 (1970).

[5] T.M. O'Neil, J.H. Winfrey and J.H. Malmberg, Nonlinear interaction of a small cold beam and a plasma, Phys. Fl. **14** 1204 (1971).

[6] R.L. Dewar, Saturation of kinetic plasma instabilities by particle trapping, Phys. Fl. **16** 431 (1973).

[7] A. Simon and M. Rosenbluth, Single-mode saturation of the bump-on-tail instability: immobile ions, Phys. Fl. **19** 1567 (1976).

[8] J. Denavit, Simulations of the single-mode, bump-on-tail instability, Phys. Fl. **28** 2773 (1985).

[9] A. Simon, S. Radin, and R.W. Short, Long-time simulation of the single-mode bump-on-tail instability, Phys. Fl. **31** 3649 (1988).

[10] P. Janssen and J. Rasmussen, Limit cycle behavior of the bump-on-tail instability, Phys. Fl. **24** 268 (1981).

[11] C. Burnap, M. Miklavcic, B. Willis and P. Zweifel, Single-mode saturation of a linearly unstable plasma, Phys. Fl. **28** 110 (1985).

[12] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, (Springer-Verlag, New York, 1983).

[13] J.D. Crawford, Introduction to bifurcation theory, Rev. Mod. Phys. **63** 991-1037 (1991).

[14] P.J. Morrison, The Maxwell-Vlasov equations as a continuous Hamiltonian system, Phys. Lett. A **80** 383-386 (1980).

[15] J.E. Marsden and A. Weinstein, The Hamiltonian structure of the Maxwell-Vlasov equations, Physica D **4** 394-406 (1982).

[16] D.D. Holm, J.E. Marsden, T. Ratiu, and A. Weinstein, Nonlinear stability of fluid and plasma equilibria, Phys. Reports **123** 1-116 (1985).

[17] J.D. Crawford and P. Hislop, Application of the method of spectral deformation to the Vlasov-Poisson model, Ann. Phys. **189** 265-317 (1989).

[18] J.D. Crawford, Universal trapping scaling on the unstable manifold for an unstable electrostatic mode, Phys. Rev. Lett. **73** 656-659 (1994).

[19] S.I. Tsunoda, F. Doveil and J.H. Malmberg, Nonlinear interaction between a warm electron beam and a single wave, Phys. Rev. Lett. **59** 2752 (1987).

[20] In addition to the lack of consensus on the correct scaling, these authors are often unaware of previous work. From the viewpoint of this paper, the most important theoretical discussion is Baldwin's analysis [2], and he is not referenced in any of the subsequent publications. [3–11] Baldwin finds the pinching singularity at lowest nonlinear order and correctly notes the effect this singularity will have on the saturation level of the mode. He does not appear to recognize the complete generality of this result, and argues that it can be reconciled with the results of Frieman *et al.* [1] if the second velocity derivative of $F_0(v, \mu_c)$ vanishes. But this is not the essential discrepancy between Baldwin's calculation and the analysis by Frieman *et al.*, rather the latter authors apply the Plemelj formula incorrectly and thereby miss the pinching singularity that Baldwin finds a year

later.

[21] J.E. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications*, (Springer-Verlag, New York, 1976). pp. 1-135, 250-304.

[22] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840** (Springer-Verlag, New York, 1981).

[23] J. Carr, *The Centre Manifold Theorem and its Applications*, (Springer-Verlag, New York, 1983).

[24] S-N. Chow and K. Lu, Invariant manifolds for flows in Banach spaces, J. Diff. Eqns. **74** 285-317 (1988).

[25] A. Mielke, Locally invariant manifolds for quasilinear parabolic equations, Rocky Mountain J. Math. **21** 707-714 (1991).

[26] M. Renardy, A centre manifold theorem for hyperbolic PDE's, Proc. Roy. Soc. Edin **122A** 363-377 (1992).

[27] A. Vanderbauwhede and G. Iooss, 1992, Centre manifold theory in infinite dimensions, *Dynamics Reported*, New Series, Vol. 1, Springer-Verlag, New York, 125-163.

[28] N.G. van Kampen, On the theory of stationary waves in plasmas, Physica **21** (1955) 949; see also N.G. van Kampen and B.U. Felderhof, *Theoretical Methods in Plasma Physics*, (North-Holland, Amsterdam, 1967).

[29] K. Case, Plasma oscillations, Ann. Phys. **7** (1959) 349-364; also Phys. Fl. **21** 249-257 (1978).

[30] S.W.H. Cowley, Growing plasma oscillations for symmetrical douple-humped velocity distributions, J. Plasma Phys. **4** 297 (1970).

[31] N.I. Muskhelishvili, *Singular Integral Equations*, (Dover, New York, 1992). pp 42-43.

[32] I do not reserve this notation for the specific choice of a Maxwellian F_0; $\epsilon_k(z)$ denotes the analytic continuation for an arbitrary equilibrium.

[33] S. Ichimaru, *Basic Principles of Plasma Physics*, (Benjamin-Cummings, Reading, MA, 1973). pp. 43-46.

[34] B.A. Shadwick and P.J. Morrison, On neutral plasma oscillations, Phys. Lett. A **184** 277-282 (1994).

[35] By *invariant* we mean that if the initial condition $f(x, v, 0)$ belongs to the subspace (manifold), then the solution $f(x, v, t)$ remains in the subspace (manifold).

[36] This is a standard result symmetric bifurcation theory; see reference [37] for a more detailed discussion.

[37] J.D. Crawford, Amplitude expansions for instabilities in populations of globally-coupled oscillators, J. Stat. Phys **74** 1047-1084 (1994).

[38] P. Hislop and J.D. Crawford, Application of spectral deformation to the Vlasov-Poisson system II: mathematical results, J. Math. Phys. **30** 2819-2837 (1989).

[39] The occurrence of such a pinching singularity appears to have been first discovered by D. Baldwin thirty years ago. [2] Unaware of Baldwin's paper, I first found the γ^{-3} singularity of the cubic coefficient several years ago and reported it in the proceedings of a Blacksburg Transport conference. [43] This original calculation used the "leaf" representation of the Vlasov equation developed with Hislop. [54] In this representation the real coefficient of γ^{-3} is manifestly negative but has a nontrivial dependence on the critical distribution function $F_0(v, \mu_c)$.

[40] I. Bernstein, J.M. Greene, and M.D. Kruskal, Exact nonlinear plasma oscillations, Phys. Rev. **108** 546-550 (1957).

[41] E. Larsen, private communication, (1989).

[42] This may require a near-identity transformation on the (r, θ) variables to remove terms quadratic in A as described elsewhere [13].

[43] J.D. Crawford, Amplitude equations on unstable manifolds: singular behavior from neutral modes, in *Modern Mathematical Methods in Transport Theory* (Operator Theory: Advances and Applications, vol. 51), W. Greenberg and J. Polewczak, eds., Birkhauser Verlag, Basel, 1991, pp. 97-108.

[44] K. Case, Stability of inviscid plane Couette flow, Phys. Fl. **3** 143 (1960).

[45] R.J. Briggs, J.D. Daugherty, and R.H. Levy, Role of Landau damping in crossed-field electron beams and inviscid shear flow, Phys. Fl. **13** 421-432 (1970).

[46] S.M. Churilov and I.G. Shukhman, Nonlinear stability of a stratified shear flow in the regime with an unsteady critical layer, J. Fluid Mech. **194** 187-216 (1988).

[47] R. Pego and M.I. Weinstein, Eigenvalues and instabilities of solitary waves, Phil. Trans. R. Soc. Lond. A **340** 47-94 (1992).

[48] R. Pego and M.I. Weinstein, Evans function, Melnikov's integral, and solitary wave instabilities, in *Differential Equations with Applications to Mathematical Physics*, W.F. Ames, E.M. Harrell II, and J.V. Herod, eds., Academic Press, Orlando, 1993. pp. 273-286.

[49] R. Pego, P. Smereka, and M.I. Weinstein, Oscillatory instability of solitary waves in a continuum model of lattice vibrations, Nonlinearity, submitted, (1994).

[50] G. Russo and P. Smereka, Kinetic theory for bubbly flow I: collisionless case, SIAM J. Appl. Math., submitted, (1994).

[51] S. Strogatz and R. Mirollo, Stability of incoherence in a population of coupled oscillators, J. Stat. Phys. **63** 613-635 (1991).

[52] S. Strogatz, R. Mirollo and P.C. Matthews, Coupled nonlinear oscillators below the

synchronization threshold: relaxation by generalized Landau damping, Phys. Rev. Lett. **68** 2730-2733 (1992).

[53] Very recent calculations by Daido indicate that this conclusion depends on the specific form of the coupling between oscillators. See H. Daido, Generic scaling at the onset of macroscopic mutual entrainment in limit-cycle oscillators with uniform all-to-all coupling, Phys. Rev. Lett. **73** 760 (1994).

[54] J.D. Crawford and P. Hislop, Vlasov equation on a symplectic leaf, Phys. Lett. A **134** 19-24 (1988).

FIGURES

FIG. 1. Linear stability boundary for the beam-plasma instability described by a Lorentzian plasma and a Lorentzian beam as in (37) with $L = 2\pi$, $n = 0.8$, $\Delta = 0.3$, and $u_p = 0.0$. The instability occurs at the longest wavelength corresponding to $k = 1$; modes with $k = 2, 3, \ldots$ are always stable. The linear spectra for (a) stable equilibria, (b) the critical equilibrium and (c) unstable equilibria are illustrated in Fig. 2.

FIG. 2. Spectrum of $\mathcal{L}$ near criticality for the beam-plasma instability of Fig. 1. (a) The subcritical spectrum contains only the continuous spectrum and a (degenerate) eigenvalue at zero which is related to the degenerate Hamiltonian structure of the Vlasov equation. [54] The continuous spectrum coincides with the imaginary axis but is slightly thickened for ease of visualization. (b) At criticality, the conjugate pair of eigenvalues, (λ, λ^*) with $\lambda = \gamma - i\omega$, appears for the first time embedded in the continuous spectrum. (c) The supercritical spectrum shows a quadruplet of eigenvalues in addition to the continuum and zero eigenvalue.

FIG. 3. Local geometry of the unstable manifold; the equilibrium F_0 is at the origin.

FIG. 4. Conjectured spiral structure in the global unstable manifold shown in cross section with the θ coordinate suppressed. The turning points of the spiral correspond to branch points, e.g. $r = r_b$, in the mapping functions describing the manifold. The time-asymptotic state at $r = r_\infty$ is not on the branch of the manifold connected to the equilibrium at $r = 0$.

TABLES

TABLE I. Order of calculation of $h_{k,l}(v)$ and p_j from $\psi_c(v)$. The flow of calculation of the $h_{k,l}(v)$ is indicated by moving downward. From $\psi_c(v)$, $h_{0,0}$ and $h_{2,0}$ can be calculated and then p_1 determined; $h_{1,0}$ and $h_{3,0}$ are calculated next from $\{p_1, h_{0,0}, h_{2,0}\}$ and then $h_{0,1}$ and $h_{2,1}$ can be evaluated. This then determines p_2, and so forth. For $N \geq 2$, p_N requires prior calculation of $h_{k,l}$ for $0 \leq k \leq N+1$ and $0 \leq l \leq N-k+1-2(\delta_{k,0}+\delta_{k,1})$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$\cdots$
p_0		$\psi_c(v)$						
		$h_{1,0}$		$h_{3,0}$				
p_1	$h_{0,0}$	-	$h_{2,0}$					
p_2	$h_{0,1}$		$h_{2,1}$					
		$h_{1,1}$			$h_{4,0}$			
p_3	$h_{0,2}$		$h_{2,2}$	$h_{3,1}$				
		$h_{1,2}$				$h_{5,0}$		
p_4	$h_{0,3}$		$h_{2,3}$	$h_{3,2}$	$h_{4,1}$			
		$h_{1,3}$					$h_{6,0}$	
p_5	$h_{0,4}$		$h_{2,4}$	$h_{3,3}$	$h_{4,2}$	$h_{5,1}$		
$\vdots$								